

# $q, t$ -CATALAN NUMBERS AND GENERATORS FOR THE RADICAL IDEAL DEFINING THE DIAGONAL LOCUS OF $(\mathbb{C}^2)^n$

KYUNGYONG LEE AND LI LI

ABSTRACT. Let  $I$  be the ideal generated by alternating polynomials in two sets of  $n$  variables. Haiman proved that the  $q, t$ -Catalan number is the Hilbert series of the graded vector space  $M(= \bigoplus_{d_1, d_2} M_{d_1, d_2})$  spanned by a minimal set of generators for  $I$ . In this paper we give simple upper bounds on  $\dim M_{d_1, d_2}$  in terms of partition numbers, and find all bi-degrees  $(d_1, d_2)$  such that  $\dim M_{d_1, d_2}$  achieve the upper bounds. For such bi-degrees, we also find explicit bases for  $M_{d_1, d_2}$ . The main idea is to define and study a nontrivial linear map from  $M$  to a polynomial ring  $\mathbb{C}[\rho_1, \rho_2, \dots]$ .

## CONTENTS

1. Introduction	2
2. Notation	5
3. Map $\varphi$ .	5
3.1. Definition and properties of $\varphi$	5
3.2. Relation between $\varphi(D)$ and $\Delta(D)$	10
4. The upper bound of $\dim M_{d_1, d_2}$	15
4.1. A characterization of the $q, t$ -Catalan number.	15
4.2. The upper bound of $\dim M_{d_1, d_2}$ .	17
5. The lower bound of $\dim M_{d_1, d_2}$	17
5.1. A homogeneous term order, leading terms and leading monomials.	17
5.2. The theorems on the lower bound of $\dim M_{d_1, d_2}$	18
5.3. Proof of the main theorem.	19
6. The condition for the equality $\dim M_{d_1, d_2} = p(d_2, k)$ to hold	25
7. Appendix	27
7.1. Table of the $q, t$ Catalan number for $n = 7$	27
7.2. Macaulay 2 code for computing $\varphi$	27

---

Research of the first author partially supported by NSF grant DMS 0901367.

## 1. INTRODUCTION

In [5], Garsia and Haiman introduced the  $q, t$ -Catalan number  $C_n(q, t)$ , and they showed that  $C_n(q, 1)$  agrees with the  $q$ -Catalan number defined by Carlitz and Riordan [2]. To be more precise, take the  $n \times n$  square whose southwest corner is  $(0, 0)$  and northeast corner is  $(n, n)$ . Let  $\mathcal{D}_n$  be the collection of Dyck paths, i.e. lattice paths from  $(0, 0)$  to  $(n, n)$  that proceed by NORTH or EAST steps and never go below the diagonal. For any Dyck path  $\Pi$ , let  $\text{area}(\Pi)$  be the number of lattice squares below  $\Pi$  and strongly above the diagonal. Then

$$C_n(q, 1) = \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)}.$$

The  $q, t$ -Catalan number  $C_n(q, t)$  also has a combinatorial interpretation using Dyck paths. Given a Dyck path  $\Pi$ , let  $a_i(\Pi)$  be the number of squares in the  $i$ -th row that lie in the region bounded by  $\Pi$  and the diagonal. Garsia and Haglund ([3], [4]) among others showed that

$$C_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)} t^{\text{dinv}(\Pi)},$$

where

$$\text{dinv}(\Pi) := |\{(i, j) \mid i < j \text{ and } a_i(\Pi) = a_j(\Pi)\}| + |\{(i, j) \mid i < j \text{ and } a_i(\Pi) + 1 = a_j(\Pi)\}|.$$

A very natural question is to find the coefficient of  $q^{d_1} t^{d_2}$  in  $C_n(q, t)$  for each pair  $(d_1, d_2)$ , in other words, to count how many Dyck paths have the same statistics  $(\text{area}, \text{dinv})$ . It is well-known that the sum  $\text{area}(\Pi) + \text{dinv}(\Pi)$  is at most  $\binom{n}{2}$ . In this paper we find coefficients of  $q^{d_1} t^{d_2}$  when  $\binom{n}{2} - d_1 - d_2$  is relatively small.

Denote by  $p(k)$  the partition number of  $k$  and by convention  $p(0) = 1$  and  $p(k) = 0$  for  $k < 0$ . Denote by  $p(b, k)$  the partition number of  $k$  into no more than  $b$  parts, and by convention  $p(0, k) = 0$  for  $k > 0$ ,  $p(b, 0) = 1$  for  $b \geq 0$ . One of our main results is as follows.

**Theorem A.** Let  $d_1, d_2$  be non-negative integers  $d_1, d_2$  with  $d_1 + d_2 \leq \binom{n}{2}$ . Define  $k = \binom{n}{2} - d_1 - d_2$  and  $\delta = \min(d_1, d_2)$ . Then the coefficient of  $q^{d_1} t^{d_2}$  in  $C_n(q, t)$  is less than or equal to  $p(\delta, k)$ , and the equality holds if and only if one the following conditions holds:

- $k \leq n - 3$ , or
- $k = n - 2$  and  $\delta = 1$ , or
- $\delta = 0$ .

This theorem is a consequence of Theorem C. It contains [10, Theorem 6] and a result of Bergeron and Chen [1, Corollary 8.3.1] as special cases. In fact it proves [10, Conjecture 8]. We feel that the coefficient of  $q^{d_1} t^{d_2}$  for general  $k$  can also be expressed in terms of partition numbers, only that the expression might be complicated. For example, we give the following conjecture which is verified for  $6 \leq n \leq 10$ .

**Conjecture.** Let  $n, d_1, d_2, \delta, k$  be as in Theorem A. If  $n - 2 \leq k \leq 2n - 8$  and  $\delta \geq k$ , then the coefficient of  $q^{d_1} t^{d_2}$  equals

$$p(k) - 2[p(0) + p(1) + \cdots + p(k - n + 1)] - p(k - n + 2).$$

As a corollary of Theorem A, we can compute some higher degree terms of the specialization at  $t = q$ .

**Corollary B.**

$$C_n(q, q) = \sum_{k=0}^{n-3} \left( p(k) \left( \binom{n}{2} - 3k + 1 \right) + 2 \sum_{i=1}^{k-1} p(i, k) \right) q^{\binom{n}{2}-k} + (\text{lower degree terms}).$$

From the perspective of commutative algebra, the  $q, t$ -Catalan number is closely related to the graded ideal  $I$  defining the diagonal locus of  $(\mathbb{C}^2)^n$ . In [6] and [7], Haiman proved that the  $q, t$ -Catalan number is the Hilbert series of the graded vector space spanned by minimal generators for  $I$ . Blowing up the ideal  $I$  gives the well-known isospectral Hilbert scheme discovered by Haiman in his proof of the  $n!$  conjecture and the positivity conjecture for the Kostka-Macdonald coefficients [6]. A natural question, posed by Haiman [8], is to study a minimal set of generators of the ideal  $I$ . An extensive study of generators of  $I$  might lead to an explicit principalization of the ideal  $I$ .

To construct a minimal set of generators of  $I$  is difficult. However, if we focus on cases when the degree is  $\binom{n}{2} - k$  where  $k \leq n - 3$ , we can give an explicit combinatorial description for a minimal set of generators.

Now we turn to a detailed description. Fix a positive integer  $n$ . Consider  $n$ -tuples of ordered points  $\{(x_i, y_i)\}_{1 \leq i \leq n}$  in the plane  $\mathbb{C}^2$ . The set of all  $n$ -tuples forms an affine space  $(\mathbb{C}^2)^n$  with coordinate ring  $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, y_1, \dots, x_n, y_n]$ . Denote by  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$  the vector space of alternating polynomials spanned by a basis  $\{\Delta(D)\}_{D \in \mathfrak{D}_n}$  defined as follows. Denote by  $\mathbb{N}$  the set of nonnegative integers. Let  $\mathfrak{D}_n$  be the set of subsets  $D = \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$  of  $\mathbb{N} \times \mathbb{N}$ . For  $D \in \mathfrak{D}_n$ , define

$$\Delta(D) := \det \begin{bmatrix} x_1^{\alpha_1} y_1^{\beta_1} & x_1^{\alpha_2} y_1^{\beta_2} & \cdots & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & x_n^{\alpha_2} y_n^{\beta_2} & \cdots & x_n^{\alpha_n} y_n^{\beta_n} \end{bmatrix}.$$

The ideal  $I \subset \mathbb{C}[\mathbf{x}, \mathbf{y}]$  is the radical ideal that defines the locus where at least two points coincide, to be precise,

$$I = \bigcap_{1 \leq i < j \leq n} (x_i - x_j, y_i - y_j).$$

Haiman [6] has proved that  $I$  is in fact generated by  $\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ , therefore is generated by  $\{\Delta(D)\}_{D \in \mathfrak{D}_n}$ .

Finding a minimal set of generators of  $I$  is equivalent to finding a basis of  $M := I/(\mathbf{x}, \mathbf{y})I$  where  $(\mathbf{x}, \mathbf{y})$  is the maximal ideal  $(x_1, y_1, \dots, x_n, y_n)$ . Since  $M$  is naturally bi-graded with

respect to  $x$ -degree and  $y$ -degree, we can write

$$M = \bigoplus_{d_1, d_2} M_{d_1, d_2}.$$

In [7, p393], Haiman discovered the amazing fact that

$$(1.1) \quad C_n(q, t) = \sum_{d_1, d_2} t^{d_1} q^{d_2} \dim M_{d_1, d_2}.$$

Setting  $q = t = 1$ , we get

$$\dim_{\mathbb{C}} M = \frac{1}{n+1} \binom{2n}{n},$$

which is the usual Catalan number  $C_n$ .

The authors showed in [10] that, when the deficit  $k = \binom{n}{2} - d_1 - d_2$  is relatively small compared to  $n$  and  $d_1, d_2$  are not too small, an explicit basis of  $M_{d_1, d_2}$  can be constructed in one-to-one correspondence with partitions of  $k$ , by using what we call *minimal staircase forms*. However the bound of  $k$  given in [10] was by no means sharp. In this paper we find all bi-degrees  $(d_1, d_2)$  for which  $\dim M_{d_1, d_2}$  are exactly partition numbers of  $k$  into no more than  $\min(d_1, d_2)$  parts. For such bi-degrees, we also find bases for  $M_{d_1, d_2}$ .

**Theorem C.** Let  $d_1, d_2$  be non-negative integers  $d_1, d_2$  with  $d_1 + d_2 \leq \binom{n}{2}$ . Define  $k = \binom{n}{2} - d_1 - d_2$  and  $\delta = \min(d_1, d_2)$ . Then  $\dim M_{d_1, d_2} \leq p(\delta, k)$ , and the equality holds if and only if one the following conditions holds:

- $k \leq n - 3$ , or
- $k = n - 2$  and  $\delta = 1$ , or
- $\delta = 0$ .

In case the equality holds, there is an explicit construction of a basis of  $M_{d_1, d_2}$ .

The theorem is a consequence of Theorem 24 and Theorem 35. Obviously, Theorem A immediately follows from Theorem C, thanks to (1.1), a theorem of Haiman. The idea of the construction consists of two parts: the easier part is to show

$$\dim M_{d_1, d_2} \leq p(\delta, k)$$

using a new characterization of  $q, t$ -Catalan numbers; the harder part is to construct a set of  $p(\delta, k)$  linearly independent elements in  $M_{d_1, d_2}$ . It seems difficult (as least to the authors) to test directly whether a given set of elements in  $M_{d_1, d_2}$  are linearly independent. We define a map  $\varphi$  sending an alternating polynomial  $f \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$  to a polynomial ring  $\mathbb{C}[\rho_1, \rho_2, \rho_3, \dots]$ . The map has two desirable properties: (i) for many  $f$ ,  $\varphi(f)$  can be easily computed, and (ii) for each bi-degree  $(d_1, d_2)$ ,  $\varphi$  induces a morphism  $\bar{\varphi} : M_{d_1, d_2} \longrightarrow \mathbb{C}[\rho_1, \rho_2, \dots]$  of  $\mathbb{C}$ -modules. Then we use the fact the linear dependency is easier to check in  $\mathbb{C}[\rho_1, \rho_2, \dots]$  than in  $M_{d_1, d_2}$ . This idea is motivated by our earlier work [10].

The structure of the paper is as follows. After introducing some notations in §2, we define and study the map  $\varphi$  in §3, then in §4 and §5 we give the upper bound and the lower bound of  $\dim M_{d_1, d_2}$ , and prove the main result in §6. For readers' convenience, we give

the table of  $q, t$ -Catalan numbers for  $n = 7$  in appendix §7.1 and a Macaulay 2 code for computing the map  $\varphi$  in §7.2.

*Acknowledgements.* We are grateful to François Bergeron, Mahir Can, Jim Haglund, Nick Loehr, Alex Woo and Alex Yong for valuable discussions and correspondence.

## 2. NOTATION

- We adopt the convention that  $\mathbb{N}$  is the set of natural numbers *including* zero, and  $\mathbb{N}^+$  is the set of positive integers.
- For  $n \in \mathbb{N}^+$ , define  $\mathfrak{D}_n = \{D = \{(a_1, b_1), \dots, (a_n, b_n)\} \mid a_i, b_i \in \mathbb{N}\}$ , i.e. an element of  $\mathfrak{D}_n$  is an ordered set  $D$  of  $n$  points in  $\mathbb{N} \times \mathbb{N}$ . Define  $\mathfrak{D} = \cup_{n=1}^{\infty} \mathfrak{D}_n$ . Similarly, define  $\mathfrak{D}'_n = \{D = \{(a_1, b_1), \dots, (a_n, b_n)\} \mid a_i \in \mathbb{Z}, b_i \in \mathbb{N}, a_i + b_i \geq 0\}$ . Define  $\mathfrak{D}' = \cup_{n=1}^{\infty} \mathfrak{D}'_n$ . We use  $P_i$  to denote the point  $(a_i, b_i)$ , and denote  $|P_i| = a_i + b_i$ ,  $|P_i|_x = a_i$ ,  $|P_i|_y = b_i$ . Unless otherwise specified, we assume throughout the paper that

$$(2.1) \quad P_1 < P_2 < \dots < P_n, \quad \text{for } D = \{P_1, \dots, P_n\}$$

where the order is defined as follows:

$$(a, b) < (a', b') \text{ if } a + b < a' + b', \text{ or if } a + b = a' + b' \text{ and } a < a'.$$

In particular,  $|P_1| \leq |P_2| \leq \dots \leq |P_n|$ .

- Given a monomial  $f = x_1^{\alpha_1} y_1^{\beta_1} \dots x_n^{\alpha_n} y_n^{\beta_n} \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ , we call  $(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i)$  the bi-degree of  $f$ . A polynomial in  $\mathbb{C}[\mathbf{x}, \mathbf{y}]$  is bi-homogeneous of bi-degree  $(d_1, d_2)$  if all its monomials have the same bi-degree  $(d_1, d_2)$ . Given  $D = \{(a_1, b_1), \dots, (a_n, b_n)\} \in \mathfrak{D}_n$ , we call  $(\sum_{i=1}^n a_i, \sum_{i=1}^n b_i)$  the bi-degree of  $D$ , which is the same as the bi-degree of the polynomial  $\Delta(D)$ .
- Let  $k, b \in \mathbb{N}^+$ . Denote the set of partitions of  $k$  as

$$\Pi_k = \{\nu = (\nu_1, \nu_2, \dots) \mid \nu_i \in \mathbb{N}^+, \nu_1 \leq \nu_2 \leq \dots, \text{ and } \nu_1 + \nu_2 + \dots = k\}.$$

Denote by  $\Pi_{b,k}$  the set of partitions of  $k$  into at most  $b$  parts.

Define the partition numbers  $p(k) = \#\Pi_k$  and  $p(b, k) = \#\Pi_{b,k}$ . By convention  $p(0) = 0$ ,  $p(0, k) = 0$  for  $k > 0$ ,  $p(b, 0) = 1$  for all  $b \geq 0$ .

- Let  $\mathbb{Z}[\rho] = \mathbb{Z}[\rho_1, \rho_2, \dots]$  be the polynomial ring with countably many variables  $\rho_1, \rho_2, \dots$ . By convention we assume  $\rho_0 = 1$ . For a partition  $\nu = (\nu_1, \nu_2, \dots) \in \Pi_k$ , define  $\rho_\nu = \rho_{\nu_1} \rho_{\nu_2} \dots \in \mathbb{Z}[\rho]$ .
- For  $n \in \mathbb{N}^+$ , denote by  $S_n$  the permutation group of  $\{1, \dots, n\}$ .
- Given two bi-homogeneous polynomial  $f, g$  of bi-degree  $(d_1, d_2)$ , let  $\bar{f}, \bar{g}$  be the corresponding element in  $M_{d_1, d_2}$ . We say that  $f \equiv g$  (modulo lower degrees) if  $\bar{f} = \bar{g}$  in  $M_{d_1, d_2}$ .

## 3. MAP $\varphi$ .

**3.1. Definition and properties of  $\varphi$ .** In this subsection we define and study the map  $\varphi$  which naturally arises when we look for a minimal set of generators of the ideal  $I$  of

alternating polynomials. For readers' convenience, a Macaulay 2 code for computing  $\varphi$  is put in Appendix.

**Definition 1.** (a) Define the map  $\varphi : \mathfrak{D}'_n \rightarrow \mathbb{Z}[\rho]$  as follows. Let  $D = \{(a_1, b_1), \dots, (a_n, b_n)\} \in \mathfrak{D}'_n$ ,  $k = \binom{n}{2} - \sum_{i=1}^n (a_i + b_i)$ , and define

$$\varphi(D) := (-1)^k \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n \left( \sum \rho_{w_1} \rho_{w_2} \cdots \rho_{w_{b_i}} \right),$$

where  $(w_1, \dots, w_{b_i})$  in the sum  $\sum \rho_{w_1} \rho_{w_2} \cdots \rho_{w_{b_i}}$  runs through the set

$$(3.1) \quad \{(w_1, \dots, w_{b_i}) \in \mathbb{N}^{b_i} \mid w_1 + \dots + w_{b_i} = \sigma(i) - 1 - a_i - b_i\},$$

with the convention that

$$\sum \rho_{w_1} \cdots \rho_{w_{b_i}} = \begin{cases} 0 & \text{if } \sigma(i) - 1 - a_i - b_i < 0; \\ 0 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i > 0; \\ 1 & \text{if } b_i = 0 \text{ and } \sigma(i) - 1 - a_i - b_i = 0. \end{cases}$$

(b) Here is an equivalent definition of  $\varphi(D)$ . Define the weight of  $\rho_i$  to be  $i$  for  $i \in \mathbb{N}^+$  and define the weight of  $\rho_0 = 1$  to be 0. Naturally the weight of any monomial  $c\rho_{i_1} \cdots \rho_{i_n}$  ( $c \in \mathbb{Z}$ ) is defined to be  $i_1 + \dots + i_n$ . For  $w \in \mathbb{N}$  and a power series  $f \in \mathbb{Z}[[\rho_1, \rho_2, \dots]]$ , denote by  $\{f\}_w$  the sum of terms of weight- $w$  in  $f$ , which is a polynomial. Define

$$h(b, w) := \{(1 + \rho_1 + \rho_2 + \cdots)^b\}_w, \quad b \in \mathbb{N}, w \in \mathbb{Z}.$$

Naturally  $h(b, w) = 0$  if  $w < 0$ . Also assume  $(1 + \rho_1 + \rho_2 + \cdots)^0 = 1$ . Then

$$\varphi(D) = (-1)^k \begin{vmatrix} h(b_1, -|P_1|) & h(b_1, 1 - |P_1|) & h(b_1, 2 - |P_1|) & \cdots & h(b_1, n - 1 - |P_1|) \\ h(b_2, -|P_2|) & h(b_2, 1 - |P_2|) & h(b_2, 2 - |P_2|) & \cdots & h(b_2, n - 1 - |P_2|) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(b_n, -|P_n|) & h(b_n, 1 - |P_n|) & h(b_n, 2 - |P_n|) & \cdots & h(b_n, n - 1 - |P_n|) \end{vmatrix}.$$

(c) Let  $D_1, \dots, D_\ell \in D'$  be of the same bi-degree and  $\sum_{i=1}^\ell c_i D_i$  be the formal sum for any  $c_i \in \mathbb{C}$  ( $1 \leq i \leq \ell$ ). Define

$$\varphi\left(\sum_{i=1}^\ell c_i D_i\right) := \sum_{i=1}^\ell c_i \varphi(D_i).$$

For any bi-homogeneous alternating polynomials  $f = \sum_{i=1}^\ell c_i \Delta(D_i) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon$ , we define

$$\varphi(f) := \varphi\left(\sum_{i=1}^\ell c_i D_i\right) = \sum_{i=1}^\ell c_i \varphi(D_i)$$

by abuse of notation. □

Before relating  $\varphi(D)$  with  $\Delta(D)$ , we shall first look at some properties of the map  $\varphi$ .

**Lemma 2.** Let  $n \in \mathbb{N}^+$ ,  $D = \{P_1, \dots, P_n\} \in \mathfrak{D}'_n$  where  $P_1 < \dots < P_n$  as in the assumption (2.1).

(i) If  $|P_i| \geq i$  for some  $1 \leq i \leq n$ , then  $\varphi(D) = 0$ .

(ii) Let  $m \in \mathbb{N}^+$  and  $Q_1, \dots, Q_m \in \mathbb{Z} \times \mathbb{N}$  satisfy  $|Q_i| = i - 1$  for  $1 \leq i \leq m$ . Let

$$\tilde{D} = \{Q_1, \dots, Q_m, P_1 + (m, 0), P_2 + (m, 0), \dots, P_n + (m, 0)\}.$$

Then  $\varphi(\tilde{D}) = \varphi(D)$ .

(iii) Let  $t \in \mathbb{N}^+$ ,  $Q = (-t, t)$  and  $\tilde{D} = \{P_1 + Q, P_2 + Q, \dots, P_n + Q\}$ . Then

$$\varphi(\tilde{D}) = \varphi(D).$$

(iv) Let  $S = \{i \mid |P_i| := a_i + b_i = i - 1\} = \{i_1 < \dots < i_\ell\}$  and assume  $i_1 = 1$ . We define the set  $\{P_{i_r}, \dots, P_{i_{r+1}-1}\}$  the  $r$ -th block of  $D$  for  $1 \leq r \leq \ell$  (assuming  $P_{i_{\ell+1}} = n + 1$ ). Then

$$\varphi(D) = \prod_{r=1}^{\ell} \varphi(\{P_{i_r} - (i_r - 1, 0), P_{i_{r+1}} - (i_r - 1, 0), \dots, P_{i_{r+1}-1} - (i_r - 1, 0)\}).$$

(v) Suppose  $|P_i| = 0$  for  $1 \leq i \leq n$ . Then  $\varphi(D) = c \cdot \rho_1^{\binom{n}{2}}$  for a positive integer  $c$ . In fact,

$$c = \frac{\prod_{i < j} (b_i - b_j)}{1!2! \cdots (n-1)!}.$$

(vi) For  $s \in \mathbb{N}^+$ , let  $D = \{(-1, 1), (0, 0), (1, 0), \dots, (s-1, 0)\}$ . Then

$$\varphi(D) = \rho_s.$$

Before giving the proof, let us look at some examples explaining the lemma.

**Example 3.** (i) We have  $\varphi(\{(0, 0), (1, 0), (2, 1), (3, 0)\}) = 0$  since  $|P_3| = 2 + 1 = 3$ .

(ii) Let  $D = \{(-1, 1), (0, 0), (0, 1)\}$ ,  $m = 2$ ,  $Q_1 = (0, 0)$ ,  $Q_2 = (1, 0)$ . Then

$$\varphi(\{(0, 0), (1, 0), (1, 1), (2, 0), (2, 1)\}) = \varphi(D),$$

i.e.

$$\varphi(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}) = \varphi(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}).$$

(iii) Let  $D = \{(0, 0), (0, 1), (1, 0)\}$ ,  $t = 1$ . Then

$$\varphi(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}) = \varphi(\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}).$$

(iv) Let  $D = \{P_1, P_2, P_3, P_4, P_5, P_6\} = \{(0, 0), (0, 1), (1, 0), (2, 1), (3, 0), (2, 2)\}$ . There are 3 blocks in  $D$ , namely  $\{P_1\}$ ,  $\{P_2, P_3\}$  and  $\{P_4, P_5, P_6\}$ . Then

$$\varphi(D) = \varphi(\{(0, 0)\}) \cdot \varphi(\{(-1, 1), (0, 0)\}) \cdot \varphi(\{(-1, 1), (0, 0), (-1, 2)\}),$$

i.e.

$$\varphi(\text{diagram}) = \varphi(\text{diagram}_1) \cdot \varphi(\text{diagram}_2) \cdot \varphi(\text{diagram}_3).$$

(v) For  $D = \{(-n+1, n-1), (-n+2, n-2), \dots, (-1, 1), (0, 0)\}$ ,  $\varphi(D) = \rho_1^{\binom{n}{2}}$ .

*Proof of Lemma 2.* (i) It immediately follows from the condition (3.1).

(ii) By definition,

$$\varphi(\tilde{D}) = (-1)^{\tilde{k}} \sum_{\tilde{\sigma} \in S_{m+n}} \text{sgn}(\tilde{\sigma}) \prod_{i=1}^{m+n} \left( \sum \rho_{w_1} \cdots \rho_{w_{b_i}} \right),$$

where  $w_1, \dots, w_{b_i} \in \mathbb{N}$  and

$$w_1 + \dots + w_{b_i} = \tilde{\sigma}(i) - 1 - a_i - b_i = \begin{cases} \tilde{\sigma}(i) - i, & \text{if } i \leq m; \\ \tilde{\sigma}(i) - 1 - m - |P_{i-m}|, & \text{if } i > m. \end{cases}$$

If  $\tilde{\sigma}(i) < i$  for some  $i \leq m$ , then no  $w_1, \dots, w_{b_i}$  satisfies the condition,  $\prod_{i=1}^{m+n} (\sum \rho_{w_1} \cdots \rho_{w_{b_i}}) = 0$ , hence the summand corresponding to  $\tilde{\sigma}$  does not contribute to  $\varphi(\tilde{D})$ . So we only need to consider those  $\tilde{\sigma}$  satisfying  $\tilde{\sigma}(i) = i$  ( $1 \leq i \leq m$ ). Each such  $\tilde{\sigma}$  corresponds to a permutation of  $\{m+1, \dots, m+n\}$ , and by translation, a permutation of  $\{1, \dots, n\}$ . To be precise,

$$\sigma(i-m) = \tilde{\sigma}(i) - m, \quad m+1 \leq i \leq m+n.$$

Then  $\tilde{\sigma}(i) - 1 - m - |P_{i-m}| = \sigma(i-m) - 1 - |P_{i-m}|$  for  $m+1 \leq i \leq m+n$ . Moreover,

$$\tilde{k} = \binom{n+m}{2} - \sum_{i=1}^m |Q_i| - \sum_{i=1}^n (|P_i| + m) = \binom{n}{2} - \sum_{i=1}^n |P_i| = k.$$

Comparing with the definition of  $\varphi(D)$ , we conclude that  $\varphi(\tilde{D}) = \varphi(D)$ .

(iii) It suffices to prove the case when  $t = 1$ . Define

$$\mathbf{v}_i = \begin{bmatrix} h(b_1, i - |P_1|) \\ h(b_2, i - |P_2|) \\ \vdots \\ h(b_n, i - |P_n|) \end{bmatrix}, \quad \mathbf{v}'_i = \begin{bmatrix} h(b_1 + 1, i - |P_1|) \\ h(b_2 + 1, i - |P_2|) \\ \vdots \\ h(b_n + 1, i - |P_n|) \end{bmatrix}, \quad 0 \leq i \leq n-1.$$

By the definition of the map  $\varphi$ ,

$$\varphi(D) = (-1)^k \det(\mathbf{v}_0, \dots, \mathbf{v}_{n-1}), \quad \varphi(\tilde{D}) = (-1)^k \det(\mathbf{v}'_0, \dots, \mathbf{v}'_{n-1}).$$

By the definition of the function  $h$ , it is easy to deduce the relation

$$h(b+1, w) = h(b, w) + \rho_1 h(b, w-1) + \rho_2 h(b, w-2) + \dots.$$

Since  $|P_1|, \dots, |P_n|$  are non-negative integers, the above relation implies

$$\mathbf{v}'_i = \mathbf{v}_i + \rho_1 \mathbf{v}_{i-1} + \rho_2 \mathbf{v}_{i-2} + \dots + \rho_i \mathbf{v}_0, \quad 0 \leq i \leq n-1,$$

hence

$$\varphi(D) = (-1)^k \det(\mathbf{v}_0, \dots, \mathbf{v}_{n-1}) = (-1)^k \det(\mathbf{v}'_0, \dots, \mathbf{v}'_{n-1}) = \varphi(\tilde{D}).$$



(iv) Suppose the summand in  $\varphi(D)$  corresponding to  $\sigma \in S_n$  does contribute. By the definition of  $\varphi(D)$ , it is necessary that  $\sigma(j) - 1 - |P_j| \geq 0$  for  $1 \leq j \leq n$ . Let  $1 \leq r \leq \ell$ . For  $j \geq i_r$ , we have  $|P_j| \geq |P_{i_r}|$ , therefore

$$\sigma(j) \geq 1 + |P_j| \geq 1 + |P_{i_r}| = i_r.$$

So  $\sigma$  maps the set  $\{i_r, i_r + 1, \dots, n\}$  to itself for every  $r$ . It follows that  $\sigma$  maps each block to itself. Let  $\sigma_r$  be the restriction of  $\sigma$  to  $\{i_r, i_r + 1, \dots, i_{r+1} - 1\}$ . Define  $n_r = i_{r+1} - i_r$ ,  $k_r = \sum_{j=i_r-1}^{i_{r+1}-2} j - \sum_{j=i_r}^{i_{r+1}-1} |P_j|$ . Then by (ii) and a routine computation, we have

$$\begin{aligned} \varphi(D) &= (-1)^{k_1 + \dots + k_\ell} \sum_{\sigma_1, \dots, \sigma_\ell} \text{sgn}(\sigma_1) \cdots \text{sgn}(\sigma_\ell) \prod_{i=1}^{n_1 + \dots + n_\ell} \left( \sum \rho_{w_1} \cdots \rho_{w_{b_i}} \right) \\ &= \prod_{r=1}^{\ell} \left( (-1)^{k_r} \sum_{\sigma_r} \text{sgn}(\sigma_r) \prod_{i=1}^{n_r} \left( \sum \rho_{w_1} \cdots \rho_{w_{b_i}} \right) \right) \\ &= \prod_{r=1}^{\ell} \varphi(\{P_{i_r} - (i_r - 1, 0), P_{i_{r+1}} - (i_r - 1, 0), \dots, P_{i_{r+1}-1} - (i_r - 1, 0)\}). \end{aligned}$$

(v) We rewrite the definition of  $\varphi$  as

$$(3.2) \quad \varphi(D) = (-1)^k \sum_{(\sigma, \{w_j^{(i)}\})} \left( \text{sgn}(\sigma) \prod_{i=1}^n \rho_{w_1^{(i)}} \rho_{w_2^{(i)}} \cdots \rho_{w_{b_i}^{(i)}} \right),$$

where  $\{w_j^{(i)}\}$  is a set of nonnegative integers,  $1 \leq i \leq n$ ,  $1 \leq j \leq b_i$ . For  $1 \leq i \leq n$ , since  $|P_i| = 0$ , those  $w_j^{(i)}$  satisfy the condition

$$w_1^{(i)} + \cdots + w_{b_i}^{(i)} = \sigma(i) - 1.$$

Denote by  $\Sigma$  the set of all possible data  $(\sigma, \{w_j^{(i)}\})$ . Let  $\Sigma' \subset \Sigma$  be the subset consisting of those  $(\sigma, \{w_j^{(i)}\})$  such that not all  $w_j^{(i)}$  are 0 or 1. We shall define a ‘conjugation’ on the set  $\Sigma'$ , i.e. an automorphism  $f : \Sigma' \rightarrow \Sigma'$  such that  $f \circ f$  is the identity.

For  $(\sigma, \{w_j^{(i)}\}) \in \Sigma'$ , define  $m_i$  to be the number of nonzero elements in  $(w_1^{(i)}, \dots, w_{b_i}^{(i)})$ , for  $1 \leq i \leq n$ . Then

$$m_1 + \cdots + m_n \leq 0 + 1 + \cdots + (n - 1) = \binom{n}{2}.$$

Since some  $w_j^{(i)}$  is greater than 1, the inequality must be strict, therefore we can find a smallest pair  $(r, r')$  such that  $r < r'$  and  $m_r = m_{r'}$ . (Here we use the lexicographic order, i.e.,  $(r, r') < (s, s')$  if  $r < s$  or  $(r = s \text{ and } r' < s')$ .) Let

$$\{j_1 < \cdots < j_{m_r}\} := \{j \mid w_j^{(r)} \neq 0\},$$

$$\{j'_1 < \cdots < j'_{m_r}\} := \{j' \mid w_{j'}^{(r')} \neq 0\}.$$

Define  $\tilde{\sigma} = \sigma \cdot (r, r')$ , i.e.  $\tilde{\sigma}(r) = \sigma(r')$ ,  $\tilde{\sigma}(r') = \sigma(r)$ ,  $\tilde{\sigma}(\ell) = \sigma(\ell)$  for  $\ell \neq r, r'$ . Define  $\{\tilde{w}_j^{(i)}\}$  as follows. For  $i \neq r, r'$ , define  $\tilde{w}_j^{(i)} = w_j^{(i)}$  for  $1 \leq j \leq b_i$ . For  $i = r$ , define

$$\tilde{w}_{j_\ell}^{(r)} = w_{j'_\ell}^{(r')} \text{ for } 1 \leq \ell \leq m_r, \quad \text{and } \tilde{w}_j^{(r)} = 0 \text{ for } j \neq j_1, \dots, j_{m_r}.$$

Similarly for  $i = r'$ , define

$$\tilde{w}_{j'_\ell}^{(r')} = w_{j_\ell}^{(r)} \text{ for } 1 \leq \ell \leq m_r, \quad \text{and } \tilde{w}_{j'}^{(r')} = 0 \text{ for } j' \neq j'_1, \dots, j'_{m_r}.$$

Define the conjugation  $f : (\sigma, \{w_j^{(i)}\}) \mapsto (\tilde{\sigma}, \{\tilde{w}_j^{(i)}\})$ . It is immediate from the above construction that  $f$  is a conjugation. Moreover,  $f$  has no fixed point because  $\sigma \neq \tilde{\sigma}$ . Since  $\text{sgn}(\sigma) = -\text{sgn}(\tilde{\sigma})$ , the summand in (3.2) corresponding to  $(\sigma, \{w_j^{(i)}\})$  cancels with the summand corresponding to  $(\tilde{\sigma}, \{\tilde{w}_j^{(i)}\})$ .

Finally, we are left with the case when all  $w_j^{(i)}$  are 0 or 1. Using Definition 1 (b), and using the fact that the monomial  $\rho_1^w$  in  $h(b, w)$  has coefficient  $\binom{b}{w}$ , we obtain

$$\varphi(D) = (-1)^{\binom{n}{2}} \begin{vmatrix} \binom{b_1}{0} \rho_1^0 & \binom{b_1}{1} \rho_1^1 & \cdots & \binom{b_1}{n-1} \rho_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{b_n}{0} \rho_1^0 & \binom{b_n}{1} \rho_1^1 & \cdots & \binom{b_n}{n-1} \rho_1^{n-1} \end{vmatrix} = \begin{vmatrix} \binom{b_n}{0} & \binom{b_n}{1} & \cdots & \binom{b_n}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{b_1}{0} & \binom{b_1}{1} & \cdots & \binom{b_1}{n-1} \end{vmatrix} \rho_1^{\binom{n}{2}} = c \cdot \rho_1^{\binom{n}{2}},$$

where  $c$  is the second determinant. Notice that  $\binom{b}{i} = b(b-1) \cdots (b-i+1)/i!$  is a polynomial of  $b$  of degree  $i$  whose leading term is  $b^i/i!$ . By appropriate column operations, i.e., adding appropriate multiples of the first  $i-1$  columns to the  $i$ -th column for  $1 \leq i \leq n$ , we obtain

$$c = \begin{vmatrix} \binom{b_n}{0} & \binom{b_n}{1} & \cdots & \binom{b_n}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{b_1}{0} & \binom{b_1}{1} & \cdots & \binom{b_1}{n-1} \end{vmatrix} = \begin{vmatrix} 1 & b_n & \frac{b_n^2}{2!} & \cdots & \frac{b_n^{n-1}}{(n-1)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & b_1 & \frac{b_1^2}{2!} & \cdots & \frac{b_1^{n-1}}{(n-1)!} \end{vmatrix} = \frac{\prod_{i < j} (b_i - b_j)}{1!2! \cdots (n-1)!},$$

by using standard results of Vandermonde matrices. Since  $b_1 > b_2 > \cdots > b_n$  are distinct integers by assumption,  $c$  is a strictly positive integer.

(vi) Follows immediately from Definition 1 (b). □

**3.2. Relation between  $\varphi(D)$  and  $\Delta(D)$ .** We need the following elementary lemma.

**Lemma 4** ([10], Lemma 26). *For  $(\alpha_i, \beta_i) \in \mathbb{N} \times \mathbb{N}$  ( $1 \leq i \leq n$ ) and  $c, e \in \mathbb{N}$ ,*

$$\left( \sum_{i=1}^n x_i^c y_i^e \right) \cdot \begin{vmatrix} x_1^{\alpha_1} y_1^{\beta_1} & x_1^{\alpha_2} y_1^{\beta_2} & \cdots & x_1^{\alpha_n} y_1^{\beta_n} \\ x_2^{\alpha_1} y_2^{\beta_1} & x_2^{\alpha_2} y_2^{\beta_2} & \cdots & x_2^{\alpha_n} y_2^{\beta_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & x_n^{\alpha_2} y_n^{\beta_2} & \cdots & x_n^{\alpha_n} y_n^{\beta_n} \end{vmatrix} = \sum_{i=1}^n \begin{vmatrix} x_1^{\alpha_1} y_1^{\beta_1} & \cdots & x_1^{\alpha_i+c} y_1^{\beta_i+e} & \cdots & x_1^{\alpha_n} y_1^{\beta_n} \\ x_2^{\alpha_1} y_2^{\beta_1} & \cdots & x_2^{\alpha_i+c} y_2^{\beta_i+e} & \cdots & x_2^{\alpha_n} y_2^{\beta_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & \cdots & x_n^{\alpha_i+c} y_n^{\beta_i+e} & \cdots & x_n^{\alpha_n} y_n^{\beta_n} \end{vmatrix}.$$

As a consequence,

$$0 \equiv \sum_{i=1}^n \Delta(\{(\alpha_1, \beta_1), \dots, (\alpha_{i-1}, \beta_{i-1}), (\alpha_i + c, \beta_i + e), (\alpha_{i+1}, \beta_{i+1}), \dots, (\alpha_n, \beta_n)\})$$

modulo lower degrees.

Let us recall the definition of minimal staircase forms defined in [10], and then define special minimal staircase forms.

**Definition 5.** We call  $D = \{P_1, \dots, P_n\} \in \mathfrak{D}_n$  a *minimal staircase form* if  $|P_i| = i - 1$  or  $i - 2$  for every  $1 \leq i \leq n$ . For a minimal staircase form  $D$ , let  $\{i_1 < i_2 < \dots < i_\ell\}$  be the set of  $i$ 's such that  $|P_i| = i - 1$ , we define the *partition type* of  $D$  to be the partition of  $\binom{n}{2} - \sum |P_i|$  consisting of all the positive integers in the sequence

$$(i_1 - 1, i_2 - i_1 - 1, i_3 - i_2 - 1, \dots, i_\ell - i_{\ell-1} - 1, n - i_\ell).$$

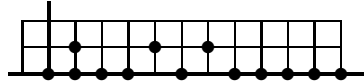
**Example 6.** For  $n = 8$  and  $D = \{P_1, \dots, P_8\}$  satisfying  $(|P_1|, \dots, |P_8|) = (0, 1, 1, 2, 4, 4, 5, 6)$ , the set  $\{i \mid |P_i| = i - 1\}$  equals  $\{1, 2, 5\}$ . The positive integers in the sequence  $(1 - 1, 2 - 1 - 1, 5 - 2 - 1, 8 - 5)$  is  $(2, 3)$ , so the partition type of  $D$  is  $(2, 3)$ .

**Definition 7.** The data  $(m, n, (r_1, \dots, r_m), (s_1, \dots, s_m)) \in \mathbb{N} \times (\mathbb{N}^+) \times \mathbb{N}^m \times \mathbb{N}^m$  satisfying  $1 \leq r_1 < r_2 < \dots < r_m < r_{m+1} := n$  and  $0 \leq s_i \leq r_{i+1} - r_i - 1$  ( $1 \leq i \leq m$ ) determines a  $D \in \mathfrak{D}_n$  as follows.

$$D = \{(0, 0), (1, 0), \dots, (n - 1, 0)\} \cup \{(r_1 - 1, 1), (r_2 - 1, 1), \dots, (r_m - 1, 1)\} \\ \setminus \{(r_1 + s_1, 0), (r_2 + s_2, 0), \dots, (r_m + s_m, 0)\}.$$

We call  $D$  a *special minimal staircase form*.

**Remark 8.** It is easy to see that a special minimal staircase form is indeed a minimal staircase form. Using the notation in the definition, the partition type of a special minimal staircase form  $D$  is obtained from  $(s_1, s_2, \dots, s_m)$  by eliminating 0's and sorting the sequence if necessary. The following picture gives a typical example of a special minimal staircase form,



where  $m = 3$ ,  $n = 13$ ,  $(r_1, r_2, r_3) = (2, 5, 7)$ ,  $(s_1, s_2, s_3) = (2, 1, 5)$ , and the partition type is  $(1, 2, 5)$ .

Let us recall the following two facts proved in [10].

**Lemma 9** (Minors Permuting Lemma in [10]). *Let  $D = \{P_1, \dots, P_n\} \in \mathfrak{D}$ ,  $h, \ell, m \in \mathbb{N}^+$  satisfy  $2 \leq h < h + \ell + m \leq n + 1$ ,  $|P_h| = h - 1$ ,  $|P_{h+\ell}| = h + \ell - 1$ ,  $|P_{h+\ell+m}| = h + \ell + m - 1$  (this condition holds if  $h + \ell + m = n + 1$  by assumption). Suppose  $|P_{h+\ell}|_x, \dots, |P_{h+\ell+m-1}|_x \geq \ell$ . Define*

$$D' = \{P_1, P_2, \dots, P_{h-1}, P_{h+\ell} - (\ell, 0), P_{h+\ell+1} - (\ell, 0), \dots, P_{h+\ell+m-1} - (\ell, 0), \\ P_h + (m, 0), P_{h+1} + (m, 0), \dots, P_{h+\ell-1} + (m, 0), P_{h+\ell+m}, \dots, P_n\}.$$

Then  $\Delta(D) \equiv \Delta(D')$  (modulo lower degrees).

**Lemma 10** (Main Theorem of [10]). *Suppose  $k$  is a positive integer such that  $n \geq 8k + 5$  and  $d_1, d_2 \geq (2k + 1)n$  are two integers whose sum is  $n(n - 1)/2 - k$ . Then  $M_{d_1, d_2}$  is minimally*

generated by  $p(k)$  elements, i.e.,  $\dim M_{d_1, d_2} = p(k)$ . Furthermore, there is a one-to-one correspondence between partitions of  $k$  and generators, namely

$$(\mu = \sum m_i j_i \in \Pi_k) \longleftrightarrow (\text{a minimal staircase form of partition type } \mu).$$

The following lemma is essential for this paper.

**Lemma 11.** (i) Let  $d_1, d_2, k \in \mathbb{N}$  and  $d_1 + d_2 = \binom{n}{2} - k$ . Define

$$\Pi'_k = \left\{ \mu \in \Pi_k \left| \begin{array}{l} \text{there exists a minimal staircase form } F_\mu \in \mathfrak{D}_n \text{ of} \\ \text{partition type } \mu \text{ and of bi-degree } (d_1, d_2) \end{array} \right. \right\}.$$

If there are coefficients  $a_\mu \in \mathbb{C}$  ( $\mu \in \Pi'_k$ ) satisfying

$$\sum_{\mu \in \Pi'_k} a_\mu \Delta(F_\mu) \equiv 0 \quad (\text{modulo lower degrees}),$$

then  $a_\mu = 0$ ,  $\forall \mu \in \Pi'_k$ . In other words,  $\{\Delta(F_\mu)\}_{\mu \in \Pi'_k}$  form a linearly independent set in  $M_{d_1, d_2}$ .

(ii) Any two special minimal staircase form in  $\mathfrak{D}_n$  of the same partition type and the same bi-degree are equivalent modulo lower degrees.

*Proof.* (i) Choose a sufficiently large integer  $N$  and choose  $(N - n)$  points  $P_{n+1}, \dots, P_N \in \mathbb{N} \times \mathbb{N}$  such that  $|P_i| = i - 1$  for  $n + 1 \leq i \leq N$  and

$$|P_{n+1}|_x + \dots + |P_N|_x \geq (2k + 1)N, \quad |P_{n+1}|_y + \dots + |P_N|_y \geq (2k + 1)N.$$

let  $F'_\mu = F_\mu \cup \{P_{n+1}, P_{n+2}, \dots, P_N\}$ . The condition of Lemma 10 is satisfied, so  $\Delta(F'_\mu)$  for  $\mu \in \Pi'_k$  are linearly independent modulo lower degrees. But  $\Delta(F'_\mu)$  is equivalent to  $\Delta(F_\mu) \cdot f_0$  for a polynomial  $f_0$  independent of  $\mu$ . (In fact  $f_0 = \prod_{i,j} a_{ij}$  for  $1 \leq i < j \leq N$  and  $j \geq n + 1$ , with appropriate choices  $a_{ij} = x_j - x_i$  or  $y_j - y_i$ . We do not need the exact formula here.) So the linear independence of  $\{\Delta(F'_\mu)\}_{\mu \in \Pi'_k}$  implies the linear independence of  $\{\Delta(F_\mu)\}_{\mu \in \Pi'_k}$ .

(ii) The claim follows immediately from Minors Permuting Lemma (Lemma 9).  $\square$

**Proposition 12.** Let  $n \in \mathbb{N}^+$ ,  $D = \{P_1, \dots, P_n\} \in \mathfrak{D}$  and  $k = \binom{n}{2} - \sum_{i=1}^n |P_i| \geq 0$ . Suppose  $N \in \mathbb{N}^+$  satisfies  $N > N_0 := (\sum_{i=1}^n |P_i|_y)(k + 1)$ . Define

$$\tilde{D} = \{(0, 0), (1, 0), \dots, (N - 1, 0), P_1 + (N, 0), \dots, P_n + (N, 0)\} \in \mathfrak{D}_{N+n}.$$

Let  $d_2 = \sum_i |P_i|_y$  be the  $y$ -degree of  $D$  (which is also the  $y$ -degree of  $\tilde{D}$ ). For  $\mu \in \Pi_{d_2, k}$ , let  $F_\mu$  be a special minimal staircase form with the same bi-degree as  $\tilde{D}$  and be of partition type  $\mu$ . Then there exist unique integers  $a_\mu$  ( $\mu \in \Pi_{d_2, k}$ ) such that

$$\Delta(\tilde{D}) \equiv \sum_{\mu \in \Pi_{d_2, k}} a_\mu \cdot \Delta(F_\mu) \quad (\text{modulo lower degrees}).$$

In fact, the integers  $a_\mu$  satisfy

$$(3.3) \quad \varphi(D) = \sum_{\mu \in \Pi_{d_2, k}} a_\mu \rho_\mu.$$

*Proof.* In this proof we use  $D \in \mathfrak{D}$  that does not satisfy the assumption (2.1).

The uniqueness of  $a_\mu$  follows from the fact that  $\{\Delta(F_\mu)\}_{\mu \in \Pi'_k}$  form a linearly independent set in  $M_{d_1, d_2}$ , proved in Lemma 11. For the existence of  $a_\mu$ , we shall give an algorithm showing that those  $a_\mu$  are exactly the integers satisfying (3.3).

We separate the set  $\{(1, 0), (2, 0), \dots, (N_0, 0)\}$  into  $(\sum_{i=1}^n |P_i|_y)$  segments, where the  $r$ -th segment for  $1 \leq r \leq (\sum_{i=1}^n |P_i|_y)$  consists of  $(k+1)$  points  $\{(i, 0) \mid (r-1)(k+1) + 1 \leq i \leq r(k+1)\}$ .

Consider the following sequence of length  $d_2$ .

( $\star$ ):  $(1, |P_1|_y), \dots, (1, 2), (1, 1), (2, |P_2|_y), \dots, (2, 2), (2, 1), \dots, (n, |P_n|_y), \dots, (n, 2), (n, 1)$  with the natural total order that  $(i', j') < (i, j)$  if  $(i', j')$  is to the left of  $(i, j)$ . For  $(i, j)$  in the above sequence, define  $r(i, j) \in \mathbb{N}^+$  to be the integer such that  $(i, j)$  is the  $r(i, j)$ -th pair in the sequence ( $\star$ ).

Denote  $Q_s^{(0)} = (s-1, 0)$  for  $1 \leq s \leq N$  and  $P_t^{(0)} = P_t + (N, 0)$  for  $1 \leq t \leq n$  and denote

$$D^{(0)} := \tilde{D} = \{Q_1^{(0)}, Q_2^{(0)}, \dots, Q_N^{(0)}, P_1^{(0)}, P_2^{(0)}, \dots, P_n^{(0)}\}.$$

Given a set of nonnegative integers  $\mathbf{w} = \{w_{j'}^{(i')}\}_{(i', j') \in (\star)}$ , we construct

$$D^{(r)} = \{Q_1^{(r)}, \dots, Q_N^{(r)}, P_1^{(r)}, \dots, P_n^{(r)}\}$$

inductively on  $r \in [1, \sum_{\ell=1}^n |P_\ell|_y]$ . Here we do not require  $D^{(r)}$  to satisfy the assumption of order defined in (2.1). Suppose  $D^{(r-1)}$  has been constructed and the  $r$ -th element in the sequence ( $\star$ ) is the pair  $(i, j)$ . Then  $D^{(r)}$  is constructed as follows.

$$\begin{aligned} Q_{(r-1)(k+1)+2+w_j^{(i)}}^{(r)} &= ((r-1)(k+1), 1); \\ Q_\ell^{(r)} &= Q_\ell^{(r-1)}, \quad \text{for } 1 \leq \ell \leq N \text{ and } \ell \neq (r-1)(k+1) + 2 + w_j^{(i)}; \\ P_i^{(r)} &= P_i^{(r-1)} + (w_j^{(i)} + 1, -1); \\ P_\ell^{(r)} &= P_\ell^{(r-1)}, \quad \text{for } 1 \leq \ell \leq n \text{ and } \ell \neq i. \end{aligned}$$

The following can be proved inductively on  $r$ :

$$(3.4) \quad \Delta(\tilde{D}) \equiv (-1)^r \sum_{\mathbf{w}} \Delta(D_{\mathbf{w}}^{(r)}), \quad (\text{modulo lower degrees})$$

where  $\mathbf{w}$  runs through all possible sets of integers  $\{w_{j'}^{(i')}\}_{(i', j') \leq (i, j)}$  where  $w_{j'}^{(i')} \in [0, k]$ . Indeed, for  $r = 1$  and  $|P_1|_y > 0$  (the case  $|P_1|_y = 0$  is similar) we need to show that (for simplicity of notation we use  $w$  in place of  $w_{|P_1|_y}^{(1)}$ )

$$\Delta(\tilde{D}) + \sum_{0 \leq w \leq k} \Delta(\{(0, 0), \dots, (w, 0), (0, 1), (w+2, 0), \dots, (N-1, 0), P_1 + (w+1, -1), P_2, \dots, P_n\})$$

is equivalent to 0 modulo lower degrees. But this is an immediate consequence of Lemma 4 by plugging in  $(c, e) = P_1^{(0)} - (0, 1)$  and

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots\} = \{(0, 0), (1, 0), \dots, (N-1, 0), (0, 1), P_2^{(0)}, P_3^{(0)}, \dots, P_n^{(0)}\}.$$

Here we can assume  $w \leq k$  because otherwise the total degree of the polynomial  $\Delta(\tilde{D})$  is strictly greater than  $\binom{N+n}{2}$  hence  $\Delta(\tilde{D}) \equiv 0$  modulo lower degrees.

For  $r = 2$ , we only consider the case  $|P_1|_y \geq 2$ , since the other case is similar. By induction we have

$$\Delta(\tilde{D}) \equiv - \sum_{0 \leq w_{|P_1|_y}^{(1)} \leq k} \Delta(D_{w_{|P_1|_y}^{(1)}}^{(1)}), \quad (\text{modulo lower degrees}).$$

By a similar argument as in the case  $r = 1$ ,

$$\Delta(D_{w_{|P_1|_y}^{(1)}}^{(1)}) \equiv - \sum_{0 \leq w_{|P_1|_y-1}^{(1)} \leq k} \Delta(D_{\{w_{|P_1|_y}^{(1)}, w_{|P_1|_y-1}^{(1)}\}}^{(1)}), \quad (\text{modulo lower degrees}).$$

Combine the above two formulas together, we have

$$\Delta(\tilde{D}) \equiv (-1)^2 \sum_{0 \leq w_{|P_1|_y}^{(1)}, w_{|P_1|_y-1}^{(1)} \leq k} \Delta(D_{\{w_{|P_1|_y}^{(1)}, w_{|P_1|_y-1}^{(1)}\}}^{(1)}), \quad (\text{modulo lower degrees}).$$

An easy induction similar to the above argument gives the proof of (3.4).

Now look at (3.4) when  $r = r_0 = \sum_{\ell=1}^n |P_\ell|_y$ . The  $y$ -coordinates of  $P_1^{(r_0)}, \dots, P_n^{(r_0)}$  are all zero. A necessary condition for  $\Delta(D_{\mathbf{w}}^{(r_0)}) \neq 0$  is that

$$\{|P_1^{(r_0)}|_x, |P_2^{(r_0)}|_x, \dots, |P_n^{(r_0)}|_x\} \text{ is a permutation of } \{N, N+1, \dots, N+n-1\}$$

and hence we can assume such a condition holds. Let  $\sigma \in S_n$  be the permutation that satisfies  $|P_i^{(r_0)}|_x = \sigma(i) + N - 1$ . Since

$$P_i^{(r_0)} = P_i^{(0)} + \sum_{j=1}^{|P_i|_y} w_j^{(i)},$$

we have

$$\sum_{j=1}^{|P_i|_y} w_j^{(i)} = P_i^{(r_0)} - P_i^{(0)} = (\sigma(i) + N - 1) - (N + |P_i|) = \sigma(i) - 1 - |P_i| = \sigma(i) - 1 - a_i - b_i,$$

which is exactly the condition in the definition of  $\varphi(D)$  (cf. Definition 1(a)). Next, we shall figure out the correct sign. For this, we have to rearrange the order of points in  $D_{\mathbf{w}}^{(r_0)}$  to satisfy the condition (2.1). For  $1 \leq r \leq \sum_{\ell=1}^n |P_\ell|_y$ , the  $r$ -th segment

$$((r-1)(k+1)+1, 0), ((r-1)(k+1)+2, 0), \dots, ((r-1)(k+1)+1+w_j^{(i)}, 0), \dots, (r(k+1), 0)$$

is modified to

$$((r-1)(k+1)+1, 0), ((r-1)(k+1)+2, 0), \dots, ((r-1)(k+1), 1), \dots, (r(k+1), 0).$$

The only change is that the point  $((r-1)(k+1)+1+w_j^{(i)}, 0)$  is replaced by  $((r-1)(k+1), 1)$ . To rearrange this segment into correct order, we need to move the  $(1+w_j^{(i)})$ -th point in front of the first point, so the change of sign is  $(-1)^{w_j^{(i)}}$ . On the other hand, rearranging

$\{P_1^{(r_0)}, \dots, P_1^{(r_0)}\}$  to the correct order incurs a sign change  $\text{sgn}(\sigma)$ . So the overall sign change is

$$(-1)^{\sum_{i=1}^n \sum_{j=1}^{|P_i|} w_j^{(i)}} \cdot \text{sgn}(\sigma) = (-1)^{\sum_{i=1}^n (\sigma(i)-1-|P_i|)} \cdot \text{sgn}(\sigma) = (-1)^k \text{sgn}(\sigma),$$

which coincides with the signs in the definition of  $\varphi(D)$  (cf. Definition 1(a)).

Finally, note that  $D_{\mathbf{w}}^{(r_0)}$  (after rearranging it to the correct order) is a special minimal staircase form defined in Definition 7. The partition type of  $D_{\mathbf{w}}^{(r_0)}$  is  $(w_j^{(i)})_{i,j}$ , which is compatible with the definition (3.2) of  $\varphi(D)$ . Thus we have finished the proof of Proposition 12.  $\square$

#### 4. THE UPPER BOUND OF $\dim M_{d_1, d_2}$

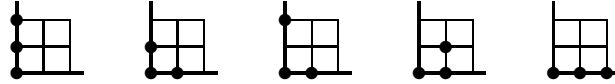
**4.1. A characterization of the  $q, t$ -Catalan number.** Recall the following conjecture we gave in [10]. We would like to point out that Mahir Can and Nick Loehr gave an equivalent conjecture in their unpublished work.

**Conjecture 13.** *Let  $\Lambda_n$  be the set of integer sequences  $\lambda_1 \geq \dots \geq \lambda_{n-1} \geq \lambda_n = 0$  satisfying  $\lambda_i \leq n - i$  for all  $i \in [1, n]$ . For any  $\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$ , let*

$$a_i = n - i - \lambda_i, \quad b_i = \#\{j \mid i < j \leq n, \lambda_i - \lambda_j + i - j \in \{0, 1\}\}$$

*and define  $D(\lambda) = \{(a_i, b_i) \mid 1 \leq i \leq n\}$ . Then  $\{\Delta(D(\lambda))\}_{\lambda \in \Lambda_n}$  generates the ideal  $I$ .*

**Example 14.** For  $n = 3$ ,  $\Lambda_3$  consists of  $(2, 1, 0), (1, 1, 0), (2, 0, 0), (1, 0, 0), (0, 0, 0)$ , the corresponding  $D(\lambda)$  are



We shall not prove the conjecture in this paper. Instead, we give a characterization of  $D(\lambda)$  appeared in the conjecture. This characterization will be used to provide an upper bound of  $\dim M_{d_1, d_2}$ .

**Definition 15.** Let  $\mathfrak{D}_n^{\text{catalan}}$  be the set consisting of  $D \subset \mathbb{N} \times \mathbb{N}$ , where  $D$  contains  $n$  points satisfying the following conditions.

(a) If  $(p, 0) \in D$  then  $(i, 0) \in D, \forall i \in [0, p]$ .

(b) For any  $p \in \mathbb{N}$ ,

$$\#\{j \mid (p+1, j) \in D\} + \#\{j \mid (p, j) \in D\} \geq \max\{j \mid (p, j) \in D\} + 1.$$

(If  $\{j \mid (p, j) \in D\} = \emptyset$ , then we require that no point  $(i, j) \in D$  satisfies  $i \geq p$ .)

**Proposition 16.** *The map  $\theta : \Lambda_n \rightarrow \mathfrak{D}_n^{\text{catalan}}$  sending  $\lambda$  to  $D(\lambda)$  is one-to-one.*

*Proof.* We first show that  $D(\lambda)$  is in  $\mathfrak{D}_n^{\text{catalan}}$ , i.e., it satisfies conditions (a)(b) of Definition 15.

By the definition of  $D(\lambda)$ , suppose  $a_i = a_{i'}$  for some  $i, i' \in [1, n]$ , then  $i \leq i'$  if and only if  $b_i \geq b_{i'}$ . Indeed, suppose  $i \leq i'$ . Since  $a_i = a_{i'}$  implies  $(\lambda_i + i) = (\lambda_{i'} + i')$ , we have

$$\{j \mid i < j \leq n, (\lambda_i + i) - (\lambda_j + j) \in \{0, 1\}\} \supseteq \{j \mid i' < j \leq n, (\lambda_{i'} + i') - (\lambda_j + j) \in \{0, 1\}\},$$

hence  $b_i \geq b_{i'}$ .

For (a), suppose  $(a_\ell, b_\ell) = (p, 0) \in D(\lambda)$  and  $(p-1, 0) \notin D(\lambda)$ . Since

$$a_i - a_{i+1} = (n - i - \lambda_i) - (n - i - 1 - \lambda_{i+1}) = 1 - (\lambda_i - \lambda_{i+1}) \leq 1, \quad \forall i \in [1, n-1]$$

and  $a_n = 0$ , there exists  $i \in [\ell+1, n]$  such that  $a_i = p-1$ . Suppose  $i_0$  is maximal among all such  $i$ . Since  $(a_\ell, b_\ell) = (p, 0)$ , we have  $a_i < p$  for all  $i > \ell$ . Therefore

$$b_{i_0} = \#\{j \mid i_0 < j \leq n, a_j - a_{i_0} \in \{0, 1\}\} = \#\{j \mid i_0 < j \leq n, a_j \in \{p-1, p\}\} = 0, \\ (a_{i_0}, b_{i_0}) = (p-1, 0), \text{ which contradicts our assumption that } (p-1, 0) \notin D(\lambda).$$

For (b), if  $\{j \mid (p, j) \in D\} = \emptyset$ , then since  $a_i - a_{i+1} \leq 1 \forall i$ , there is no point in  $D$  whose  $x$ -coordinate is greater than or equal to  $p$ . Now assume  $\{j \mid (p, j) \in D\} \neq \emptyset$ , define  $q = \max\{j \mid (p, j) \in D\}$ , and  $(a_\ell, b_\ell) = (p, q) \in D$ . By the definition of  $b_\ell$ ,

$$q = b_\ell = \#\{j \mid \ell < j \leq n, a_j - a_\ell \in \{0, 1\}\} = \#\{j \mid \ell < j \leq n, a_j = p \text{ or } p+1\},$$

therefore,

$$\#\{j \mid (p+1, j) \in D\} + \#\{j \mid (p, j) \in D\} \geq q+1.$$

So  $D(\lambda)$  is in  $\mathfrak{D}_n^{\text{catalan}}$ .

To show that  $\theta : D \mapsto D(\lambda)$  is a bijection, it suffices to construct a map  $\theta^{-1}$  sending  $D(\lambda)$  back to  $\lambda$ . We give an inductive construction on  $n$ . Let  $p \in \mathbb{N}$  be the minimal integer such that

$$\#\{j \mid (p+1, j) \in D\} + \#\{j \mid (p, j) \in D\} \leq \max\{j \mid (p, j) \in D\} + 1.$$

Let  $q = \max\{j \mid (p, j) \in D\}$  and define  $(a_1, b_1) = (p, q) \in D$ . Now  $D' := D \setminus \{(a_1, b_1)\}$  has  $(n-1)$  points and we can check that it is in  $\mathfrak{D}_{n-1}^{\text{catalan}}$ . By induction we have  $\theta^{-1}(D') = (\lambda'_1, \lambda'_2, \dots, \lambda'_{n-1})$ . Then we define

$$\theta^{-1}(D) = (n-1-p, \lambda'_1, \lambda'_2, \dots, \lambda'_{n-1}).$$

To check that it is in  $\Lambda_n$ , we need to show  $n-1-p \geq \lambda'_1$ , i.e.,  $p \leq (n-1) - \lambda'_1 = a'_1 + 1$ , where  $a'_1$  is the minimal integer that

$$\#\{j \mid (a'_1 + 1, j) \in D'\} + \#\{j \mid (a'_1, j) \in D'\} \leq \max\{j \mid (a'_1, j) \in D'\} + 1.$$

But  $D$  and  $D'$  coincide on column  $0, 1, \dots, p-1$ , therefore  $a'_1 \geq p-1$ .

To check that  $\theta$  and  $\theta^{-1}$  are inverse to each other is routine and we shall skip.  $\square$

**Remark 17.** The above proposition is also discovered independently by Alexander Woo [12].

**Corollary 18.** *The dimension of  $M_{d_1, d_2}$ , i.e., the coefficient of  $q^{d_1} t^{d_2}$  in the  $q, t$ -Catalan number  $C_n(q, t)$ , is equal to the number of  $D \in \mathfrak{D}_n^{\text{catalan}}$  such that the  $x$ -degree (resp.  $y$ -degree) of  $D$  is  $d_1$  (resp.  $d_2$ ).*



*Proof.* It is an immediate consequence of Proposition 16, by using Garsia and Haglund's description of  $q, t$ -Catalan number ([3], [4]), which asserts, in notations of Conjecture 13, that

$$C_n(q, t) = \sum_{\lambda \in \Lambda} q^{\sum a_i} t^{\sum b_i}.$$

□

**4.2. The upper bound of  $\dim M_{d_1, d_2}$ .** In order to compare  $M_{d_1, d_2}$  for different  $n$ , we use  $M_{d_1, d_2}^{(n)}$  to specify which  $n$  we are considering.

**Proposition 19.** *Let  $\ell, n \in \mathbb{N}^+$ . Then we have*

$$\dim M_{d_1, d_2}^{(n)} \leq \dim M_{d_1 + \binom{\ell}{2} + n\ell, d_2}^{(n+\ell)}.$$

*In particular, let  $k = \binom{n}{2} - d_1 - d_2$ , then*

$$\dim M_{d_1, d_2} \leq p(d_2, k).$$

*Proof.* For any  $D^{(n)} \in \mathfrak{D}_n^{\text{catalan}}$  whose bi-degree is  $(d_1, d_2)$ , we define

$$D^{(n+\ell)} = \{(0, 0), (1, 0), \dots, (\ell - 1, 0)\} \cup (D^{(n)} + (\ell, 0)),$$

where  $D^{(n)} + (\ell, 0)$  means translating the set  $D^{(n)}$  by the vector  $(\ell, 0)$ . It is easy to verify that  $D^{(n+\ell)} \in \mathfrak{D}_{n+\ell}^{\text{catalan}}$  has bi-degree  $(d_1 + \binom{\ell}{2} + n\ell, d_2)$ . By Corollary 18 we have proved the first assertion.

For any  $D^{(n)} \in \mathfrak{D}_n$  of bi-degree  $(d_1, d_2)$ , by taking sufficiently large  $\ell$  and applying Proposition 12, we get

$$\Delta(D^{(n+\ell)}) \equiv \sum_{\mu \in \Pi_{d_2, k}} a_\mu \cdot \Delta(F_\mu) \quad (\text{modulo lower degrees}),$$

where  $F_\mu \in \mathfrak{D}_{n+\ell}$  are special minimal staircase forms of bi-degree  $(d_1 + \binom{\ell}{2} + n\ell, d_2)$  and of partition type  $\mu$ . This implies

$$\dim M_{d_1 + \binom{\ell}{2} + n\ell, d_2}^{(n+\ell)} \leq p(d_2, k),$$

therefore

$$\dim M_{d_1, d_2}^{(n)} \leq \dim M_{d_1 + \binom{\ell}{2} + n\ell, d_2}^{(n+\ell)} \leq p(d_2, k).$$

□

## 5. THE LOWER BOUND OF $\dim M_{d_1, d_2}$

### 5.1. A homogeneous term order, leading terms and leading monomials.

**Definition 20.** (a) Let  $k \in \mathbb{N}^+$  and denote by  $\mathbb{Q}[\rho]_k \subset \mathbb{Q}[\rho]$  the vector space spanned by monomials  $\rho_\nu = \prod \rho_{\nu_i}$  for all sequences of positive integers  $\nu = (\nu_1 \leq \nu_2 \leq \dots \leq \nu_m)$  satisfying  $\sum \nu_i = k$ . Equivalently,  $\mathbb{Q}[\rho]_k$  is the set of weighted homogeneous polynomials of weight  $k$  by assigning the weight of  $\rho_i$  to be  $i$ ,  $\forall i \in \mathbb{N}^+$ .

(b) For any  $\nu, \mu \in \mathbb{Q}[\rho]_k$ , denoted by  $\nu = (\nu_1 \leq \nu_2 \leq \dots \leq \nu_m)$  and  $\mu = (\mu_1 \leq \mu_2 \leq \dots \leq \mu_n)$ , we define  $\rho_\nu < \rho_\mu$  if there is a positive integer  $j \leq \min(m, n)$  such that  $\nu_i = \mu_i$  for  $1 \leq i \leq j-1$  and  $\nu_j < \mu_j$ . This clearly defines a total order on  $\mathbb{Q}[\rho]_k$ ,  $\forall k \in \mathbb{N}^+$ .

(c) For  $k \in \mathbb{N}^+$  and a nonzero polynomial  $f = \sum a_\nu \rho_\nu \in \mathbb{Q}[\rho]_k$  where  $a_\nu \in \mathbb{Q}$ , we define the leading monomial of  $f$  to be the

$$\text{LM}(f) := \max\{\rho_\nu | a_\nu \neq 0\},$$

and define the leading term of  $f$  to be  $\text{LT}(f) := a_\nu \rho_\nu$  where  $\rho_\nu = \text{LM}(f)$ . For  $c \in \mathbb{Q} \setminus \{0\}$ , define  $\text{LT}(c) = 1$  and  $\text{LM}(c) = c$ .  $\square$

**Example 21.** Let  $f = 2\rho_1\rho_2\rho_7 - 5\rho_4\rho_6$ . Since  $\rho_1\rho_2\rho_7 < \rho_4\rho_6$ , we have

$$\text{LM}(f) = \rho_4\rho_6, \quad \text{LT}(f) = -5\rho_4\rho_6.$$

**Lemma 22.** (a) The total order of the monomials in  $\mathbb{Q}[\rho]_k$  defined in Definition 20 is preserved by multiplication: let  $\rho_\mu, \rho_{\mu'}, \rho_\nu$  be monomials in  $\mathbb{Q}[\rho]_k$ , then  $\rho_\mu \leq \rho_{\mu'}$  if and only if  $\rho_\mu \rho_\nu \leq \rho_{\mu'} \rho_\nu$ .

(b) Let  $\mu, \nu$  be monomials in  $\mathbb{Q}[\rho]_k$  and  $\mu', \nu'$  be monomials in  $\mathbb{Q}[\rho]_{k'}$  such that  $\rho_\mu \leq \rho_\nu$  and  $\rho_{\mu'} \leq \rho_{\nu'}$ . Then  $\rho_\mu \rho_{\mu'} \leq \rho_\nu \rho_{\nu'}$ .

(c) for  $f \in \mathbb{Q}[\rho]_k$ ,  $g \in \mathbb{Q}[\rho]_{k'}$  ( $k, k' \in \mathbb{N}^+$ ), we have

$$\text{LM}(fg) = \text{LM}(f)\text{LM}(g), \quad \text{LT}(fg) = \text{LT}(f)\text{LT}(g).$$

*Proof.* (a) Denote  $\mu = (\mu_1 \leq \mu_2 \leq \dots)$ ,  $\mu' = (\mu'_1 \leq \mu'_2 \leq \dots)$ . To show the “only if” part, suppose  $\mu \leq \mu'$ . It suffices to consider the case when we have a strict inequality  $\mu_1 < \mu'_1$ . Let  $\rho_\xi = \rho_\mu \rho_\nu$  and  $\rho_{\xi'} = \rho_{\mu'} \rho_\nu$ . Let  $\ell$  be the smallest integer satisfying  $\nu_\ell > \mu_1$ . Then

$$\begin{aligned} \xi &= (\nu_1, \dots, \nu_{\ell-1}, \mu_1, \dots), \\ \xi' &= (\nu_1, \dots, \nu_{\ell-1}, \min(\mu'_1, \nu_\ell), \dots). \end{aligned}$$

Since  $\mu_1 < \min(\mu'_1, \nu_\ell)$ , we have  $\rho_\xi < \rho_{\xi'}$  by definition and therefore  $\rho_\mu \rho_\nu < \rho_{\mu'} \rho_\nu$ . On the other hand, the “if” part immediately follows from the “only if” part.

(b) Applying (a) twice, we have  $\rho_\mu \rho_\nu \geq \rho_\mu \rho_{\nu'} \geq \rho_{\mu'} \rho_{\nu'}$ .

(c) It is an immediate consequence of (b).  $\square$

## 5.2. The theorems on the lower bound of $\dim M_{d_1, d_2}$ .

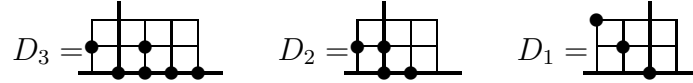
**Theorem 23.** Let  $n, k \in \mathbb{N}$ ,  $k \leq n - 4$ . Let  $d_1, d_2 \in \mathbb{N}^+$ ,  $d_1 + d_2 = \binom{n}{2} - k$ ,  $d_2 \leq d_1$ . Then for each  $\nu \in \Pi_{d_2, k}$ , there exists a  $D_\nu \in \mathfrak{D}_n$ , such that  $\Delta(D_\nu)$  has bi-degree  $(d_1, d_2)$ , and  $\text{LM}(\varphi(D_\nu)) = \rho_\nu$ .

**Theorem 24.** Let  $n, k \in \mathbb{N}$ ,  $k \leq n-3$ . Let  $d_1, d_2 \in \mathbb{N}^+$ ,  $d_1 + d_2 = \binom{n}{2} - k$ ,  $d_2 \leq d_1$ . Then for each  $\nu \in \Pi_{d_2, k}$ , there exists an alternating polynomial  $f_\nu$  of bi-degree  $(d_1, d_2)$ , either of the form  $\Delta(D)$  or of the form  $\Delta(D) - \Delta(D')$  for some  $D, D' \in \mathfrak{D}_n$ , such that  $\text{LM}(\varphi(f_\nu)) = \rho_\nu$ . Moreover,  $\dim M_{d_1, d_2} = p(d_2, k)$ , the partition number of  $k$  into at most  $d_2$  parts.

**Remark 25.** Theorem 24 gives a positive answer to Conjecture 8 in [10]. Theorem 23 and Theorem 24 are proved using the same idea. In the proofs, we give explicit constructions for  $D_\nu$  (in Theorem 23) and  $f_\nu$  (in Theorem 24). The constructions are non-canonical in the sense that there are choices to make, and it seems that no choice is more natural than others.

Before we prove the above two theorems, we shall give an example to illustrate the idea of the construction.

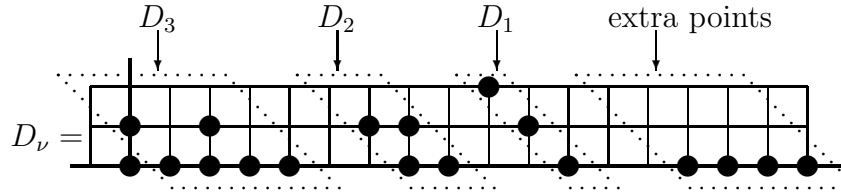
**Example 26.** We illustrate Theorem 23 by giving a construction of  $D_\nu$  for  $n = 18$ ,  $k = 14$ ,  $(d_1, d_2) = (84, 7)$ ,  $\nu = (1, 1, 1, 2, 2, 3, 4)$ . First, we divide  $\nu$  into 3 sub-partitions  $\tilde{\nu}_1 = (1, 1, 1)$ ,  $\tilde{\nu}_2 = (2, 2)$ ,  $\tilde{\nu}_3 = (3, 4)$ . For each sub-partition  $\tilde{\nu}_i$ , we construct  $D_i \in \mathfrak{D}'$  as follows:



such that in the term order defined in §5.1, the leading monomials

$$\text{LM}(\varphi(D_i)) = \rho_{\tilde{\nu}_i}, \quad \text{for } i = 1, 2, 3.$$

Now putting  $D_3, D_2, D_1$  together and adding appropriate extra points if necessary, we obtain  $D_\nu$  as in the following graph.



It satisfies  $\text{LM}(\varphi(D_\nu)) = \text{LM}(\varphi(D_1)) \cdot \text{LM}(\varphi(D_2)) \cdot \text{LM}(\varphi(D_3)) = \rho_{\tilde{\nu}_1} \rho_{\tilde{\nu}_2} \rho_{\tilde{\nu}_3} = \rho_\nu$ . □

To generalize the above example, we need to separate a partition  $\nu$  into substrings  $\tilde{\nu}_1, \tilde{\nu}_2, \dots$ , each of which contains at most 3 numbers. Every substring  $\tilde{\nu}_j$  corresponds to a  $D_j \in \mathfrak{D}'$  satisfying  $\text{LM}(\varphi(D_j)) = \rho_{\tilde{\nu}_j}$ . The correspondence is specified in table (5.2). Then by putting all  $D_j$  together and adding appropriate extra points if necessary, we obtain  $D \in \mathfrak{D}$  such that

$$\text{LM}(\varphi(D)) = \prod_j \text{LM}(\varphi(D_j)) = \prod_j \rho_{\tilde{\nu}_j} = \rho_\nu.$$

**5.3. Proof of the main theorem.** The following crucial lemma provides an effective method to verify if a set of alternating polynomials is linearly independent by using  $\varphi$ .

**Lemma 27.** Fix  $(d_1, d_2)$ . Let  $f \in \mathbb{C}[x_1, y_1, \dots, x_n, y_n]^\epsilon$  be a bi-homogeneous alternating polynomial of bi-degree  $(d_1, d_2)$ . If  $\varphi(f) \neq 0$ , then  $f \not\equiv 0$  modulo lower degrees. As a consequence,  $\varphi$  induces a well-defined linear map

$$\bar{\varphi} : M_{d_1, d_2} \longrightarrow \mathbb{C}[\rho_1, \rho_2, \dots]_k.$$

*Proof.* Suppose  $\varphi(f) \neq 0$ . By Proposition 12, after replacing  $n$  by a sufficiently large integer if necessary, we can assume that  $f$  is linearly equivalent to  $\sum_\mu a_\mu F_\mu$  modulo lower degrees, where  $F_\mu$  are special minimal staircase forms. Since  $\varphi(f) \neq 0$ , Proposition 12 guarantees  $a_\mu \neq 0$  for some  $\mu$ . Using the fact that  $\{\Delta(F_\mu)\}_\mu$  are linearly independent in  $M_{d_1, d_2}$ , we conclude that  $f \not\equiv 0$  modulo lower degrees.  $\square$

The map  $\bar{\varphi}$  is natural and useful in the study of  $M_{d_1, d_2}$ . Our main theorem (Theorem 24) implies that, for  $k := \binom{n}{2} - d_1 - d_2 \leq n - 3$  ( $d_2 \leq d_1$ ), the map  $\bar{\varphi}$  is injective and the image is spanned by  $\{\rho_\nu\}_{\nu \in \Pi_{d_2, k}}$ . For more general  $k$ , we expect that the injectivity still holds. All the computations we did so far support this conjecture.

**Conjecture 28.** The linear map  $\bar{\varphi}$  is injective.

In fact, we can show the following.

**Proposition 29.** Conjecture 13 implies Conjecture 28.

*Proof.* Assume that Conjecture 13 is true. Suppose  $f \in \mathbb{C}[x_1, y_1, \dots, x_n, y_n]^\epsilon$  is a bi-homogeneous alternating polynomial of bi-degree  $(d_1, d_2)$  satisfying  $\bar{\varphi}(f) = 0$ .

Conjecture 13 implies that the elements of  $\mathfrak{D}_n^{\text{catalan}}$  with bi-degree  $(d_1, d_2)$  form a basis of  $M_{d_1, d_2}$ , so we can express  $f$  as a linear combination  $\sum_i a_i \Delta(D_i)$ ,  $D_i \in \mathfrak{D}_n^{\text{catalan}}$ . Define  $D'_i \in \mathfrak{D}_{n+\ell}^{\text{catalan}}$  as in the proof of Proposition 19. Then

$$\bar{\varphi}\left(\sum_i a_i \Delta(D'_i)\right) = \bar{\varphi}\left(\sum_i a_i \Delta(D_i)\right) = \bar{\varphi}(f) = 0.$$

But  $\bar{\varphi} : M_{d_1+\ell, d_2} \rightarrow \mathbb{C}[\rho_1, \rho_2, \dots]_k$  is injective (since  $k \leq (n + \ell) - 3$  for sufficiently large  $\ell$ ). So  $\sum_i a_i \Delta(D'_i) = 0$ , which implies  $a_i = 0 \forall i$  and therefore  $f \equiv \sum_i a_i \Delta(D_i) = 0$ .  $\square$

**Lemma 30.** Let  $w \geq 2 \in \mathbb{N}$ . Suppose

$$D = \{P_1, \dots, P_{w+1}\} \in \mathfrak{D}'_{w+1},$$

where  $P_i$  are all distinct and

$$|P_1| = |P_2| = 0, \quad |P_i| = i - 2, \quad 3 \leq i \leq w + 1.$$

Then the leading term

$$\text{LT}(\varphi(D)) = (|P_1|_y - |P_2|_y)\rho_w.$$

In particular, the leading monomial

$$\text{LM}(\varphi(D)) = \rho_w.$$

*Proof.* Immediately follows from the definition of  $\varphi(D)$ .  $\square$

**Lemma 31.** *Let  $v, w \in \mathbb{N}$  and  $2 \leq v \leq w$ . Suppose*

$$D = \{P_1, \dots, P_{w+2}\} \in \mathfrak{D}'_{w+2},$$

*where  $P_i$  are all distinct and*

$$|P_i| = \begin{cases} 0, & \text{if } i = 1, 2; \\ i - 2, & \text{if } 3 \leq i \leq w - v + 3; \\ i - 3, & \text{if } w - v + 4 \leq i \leq w + 2. \end{cases}$$

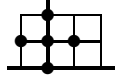
*Then the leading term*

$$\text{LT}(\varphi(D)) = -(|P_1|_y - |P_2|_y)(|P_{w-v+3}|_y - |P_{w-v+4}|_y)\rho_v\rho_w.$$

*In particular, the leading monomial*

$$\text{LM}(\varphi(D)) = \rho_v\rho_w.$$

**Example 32.** For  $v = 2, w = 3$ ,  $D = \{(-1, 1), (0, 0), (0, 1), (0, 2), (1, 1)\}$ .



A simple computation shows that

$$\varphi(D) = -\rho_2\rho_3 + \rho_1\rho_4 + \rho_1\rho_2^2 - 2\rho_1^2\rho_3 + 2\rho_1^3\rho_2 - \rho_1^5,$$

so the  $\text{LT}(\varphi(D)) = -(1 - 0)(2 - 1)\rho_2\rho_3 = -\rho_2\rho_3$  as asserted in the above lemma.

*Proof of Lemma 31.* Suppose  $\varphi(D) = \sum a_\mu \rho_\mu$ . First we show that  $a_\mu \neq 0$  implies  $\rho_\mu \leq \rho_v\rho_w$ . Suppose  $a_\mu \neq 0$ . There exist  $\sigma \in S_{w+2}$  and integers  $\{w_j^{(i)}\}$  such that the summand

$$\left( \text{sgn}(\sigma) \prod_{i=1}^n \rho_{w_1^{(i)}} \rho_{w_2^{(i)}} \cdots \rho_{w_{b_i}^{(i)}} \right)$$

in (3.2) is not zero, and

$$(5.1) \quad \rho_\mu = \prod_{i=1}^n \rho_{w_1^{(i)}} \rho_{w_2^{(i)}} \cdots \rho_{w_{b_i}^{(i)}}.$$

Because of condition (3.1), we must have

$$\sigma(i) - 1 - a_i - b_i \geq 0, \quad \forall i \in [1, w + 2],$$

in particular,

$$\sigma(w - v + 3) \geq w - v + 2, \quad \sigma(w - v + 4) \geq w - v + 2.$$

Since  $\sigma$  is a permutation,  $\sigma(w - v + 3)$  and  $\sigma(w - v + 4)$  are different from each other, hence at least one of them is greater than or equal to  $w - v + 3$ . Let  $u$  be  $w - v + 3$  or  $w - v + 4$  such that  $\sigma(u) \geq w - v + 3$ . Since  $\sigma(u) \leq w + 2$  and  $|P_u| (= a_u + b_u) = w - v + 1$ , we have

$$1 \leq \sigma(u) - 1 - |P_u| \leq v.$$

By condition (3.1),

$$w_1^{(u)} + \dots + w_{b_u}^{(u)} = \sigma(u) - 1 - a_u - b_u \in [1, v],$$

Take  $j \in \mathbb{N}^+$ ,  $1 \leq j \leq b_u$  such that  $w_j^{(u)} \neq 0$ , then  $\rho_{w_j^{(u)}}$  is a factor of  $\rho_\mu$  by (5.1). Since  $w_j^{(u)} \leq v \leq w$ ,  $\rho_\mu \leq \rho_v \rho_w$ . Therefore  $a_\mu \neq 0$  implies  $\rho_\mu \leq \rho_v \rho_w$ .

Now we show that  $a_\mu \neq 0$  for  $\mu = (v, w)$ . Assume the monomial  $\rho_v \rho_w$  appears in (5.1). By the above argument, it is necessary that

$$\sigma(u) - 1 - |P_u| = v,$$

which implies  $\sigma(u) = w + 2$ . Denote  $\delta = u - (w - v + 3) \in \{0, 1\}$ . On the other hand, since  $\sigma(1)$  and  $\sigma(2)$  cannot be 1, we may assume  $\sigma(1 + \epsilon) \neq 1$  for  $\epsilon \in \{0, 1\}$ . Then  $\sigma(1 + \epsilon) - 1 - |P_{1+\epsilon}| = w$ , hence  $\sigma(1 + \epsilon) = w + 1$ . For every positive integer  $i \leq w + 2$  that  $i \neq 1 + \epsilon$ ,  $i \neq u$ , we must have  $\sigma(i) = 1 + |P_i|$ . So  $\sigma \in S_n$  must be one of the following.

$$\sigma(i) = \begin{cases} 1, & \text{if } i = 2 - \epsilon; \\ w + 1, & \text{if } i = 1 + \epsilon; \\ i - 1, & \text{if } \epsilon + 2 \leq i \leq w - v + 2 + \delta; \\ w + 2, & \text{if } i = w - v + 3 + \delta; \\ i - 2, & \text{if } w - v + 4 + \delta \leq i \leq w + 2, \end{cases}$$

for  $(\epsilon, \delta) = (0, 0), (0, 1), (1, 0)$  or  $(1, 1)$ . By routine computation,

$\epsilon$	$\delta$	coefficient of $\rho_v \rho_w$ corresponding to $\sigma$
0	0	$- P_1 _y  P_{w-v+3} _y$
0	1	$+ P_1 _y  P_{w-v+4} _y$
1	0	$+ P_2 _y  P_{w-v+3} _y$
1	1	$- P_2 _y  P_{w-v+4} _y$

Adding the above 4 coefficients gives

$$a_\mu = a_{(v,w)} = -(|P_1|_y - |P_2|_y)(|P_{w-v+3}|_y - |P_{w-v+4}|_y) \neq 0.$$

□

**Definition 33.** To any sequence  $\nu = \{\nu_1 \leq \nu_2 \leq \dots \leq \nu_n\}$  of positive integers, we associate a sequence  $\tilde{\nu} = \{\tilde{\nu}_i\}$  of subsequences of  $\nu$ , each subsequence has the specified number of elements as follows. Denote by  $c$  the number of 1's in  $\nu$  and  $m := n - c$ .

$$\underbrace{1, 1, 1}_3; \dots; \underbrace{1, 1, 1}_3; \underbrace{1, \dots, 1}_{1,2 \text{ or } 3}; \underbrace{\nu_{c+1}, \nu_{c+2}}_2; \dots; \underbrace{\nu_{c+2\lceil \frac{m}{2} \rceil - 3}, \dots, \nu_{c+2\lceil \frac{m}{2} \rceil - 2}}_2; \underbrace{\nu_{c+2\lceil \frac{m}{2} \rceil - 1}, \dots, \nu_{c+m}}_{1 \text{ or } 2}.$$

To be precise,

$$\tilde{\nu}_i = \begin{cases} (1, 1, 1), & 1 \leq i \leq \lceil \frac{c}{3} \rceil - 1; \\ \underbrace{(1, \dots, 1)}_{c+3-3\lceil \frac{c}{3} \rceil}, & i = \lceil \frac{c}{3} \rceil; \\ (\nu_{c+2(i-\lceil \frac{c}{3} \rceil)-1}, \nu_{c+2(i-\lceil \frac{c}{3} \rceil)}), & \lceil \frac{c}{3} \rceil + 1 \leq i \leq \lceil \frac{c}{3} \rceil + \lceil \frac{m}{2} \rceil - 1; \\ (\nu_{c+2\lceil \frac{m}{2} \rceil - 1}, \dots, \nu_{c+m}), & i = \lceil \frac{c}{3} \rceil + \lceil \frac{m}{2} \rceil. \end{cases}$$

**Example 34.** If  $\nu = (9)$  then  $\tilde{\nu} = ((9))$ .

If  $\nu = (1, 1, 1, 1)$  then  $\tilde{\nu} = ((1, 1, 1), (1))$ .

If  $\nu = (1, 1, 1, 1, 10)$  then  $\tilde{\nu} = ((1, 1, 1), (1), (10))$ .

If  $\nu = (1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 5, 5)$  then  $\tilde{\nu} = ((1, 1, 1), (1, 1, 1), (1, 1), (3, 3), (5, 5))$ .

If  $\nu = (1, 1, 1, 2, 2, 2, 3, 3, 7, 7)$  then  $\tilde{\nu} = ((1, 1, 1), (2, 2), (2, 3), (3, 7), (7))$ .

*Proof of Theorem 23.* The following table is the building block of our proof. In the table below,  $|\mu|$  denotes the sum of all numbers in  $\mu$ .

$\mu$	$E_\mu \in \mathfrak{D}'$	$ \mu $	$\#E_\mu$
(1,1,1)	$\{P_1, P_2, P_3\},  P_1  =  P_2  =  P_3  = 0$	3	3
(1,1)	$\{P_1, P_2, P_3, P_4\},  P_1  =  P_2  = 0,  P_3  =  P_4  = 2$	2	4
(1)	$\{P_1, P_2\},  P_1  =  P_2  = 0$	1	2
(5.2) $(v, w)$ $2 \leq v \leq w$	$\{P_1, \dots, P_{w+2}\} \in \mathfrak{D}'_{w+2}$ , such that $ P_i  = \begin{cases} 0, & \text{if } 1 \leq i \leq 2; \\ i-2, & \text{if } 3 \leq i \leq w-v+3; \\ i-3, & \text{if } w-v+4 \leq i \leq w+2. \end{cases}$	$v+w$	$w+2$
$(w)$ $w \geq 2$	$\{P_1, \dots, P_{w+1}\}$ , such that $ P_1  =  P_2  = 0,  P_i  = i-2 \ (3 \leq i \leq w+1)$	$w$	$w+1$

We claim that, in the above table, the leading monomial  $\text{LM}(\varphi(E_\mu)) = \rho_\mu$ . Indeed, the case  $\mu = (1, 1, 1)$  or  $(1)$  follows from Lemma 2 (v); the case  $\mu = (1, 1)$  follows from Lemma 2 (iv)(v); the case  $\mu = (v, w)$  follows from Lemma 31; the case  $\mu = (w)$  follows from Lemma 30.

Let  $\tilde{\nu} = \{\tilde{\nu}_1, \dots, \tilde{\nu}_m\}$  be defined as in Definition 33. The idea of the construction of  $D_\nu$  is to take the union of translations of  $E_{\tilde{\nu}_1}, \dots, E_{\tilde{\nu}_m}$  together with some points in  $\mathbb{N} \times \mathbb{N}$  that do not affect the value of  $\varphi$ .

We consider 2 cases separately.

CASE 1:  $\tilde{\nu}_m \neq (1, 1, 1)$ .

Define translating vectors  $T_1, \dots, T_m \in \mathbb{N} \times \mathbb{N}$  as follows.  $T_m = (1, 0)$ ,

$$T_i = (1 + \#E_{\tilde{\nu}_{i+1}} + \#E_{\tilde{\nu}_{i+2}} + \dots + \#E_{\tilde{\nu}_m}, 0), \quad \forall i \in [1, m-1].$$

Define

$$n_0 = 1 + \#E_{\tilde{\nu}_1} + \#E_{\tilde{\nu}_2} + \dots + \#E_{\tilde{\nu}_m}.$$

Then  $n_0 \leq (1 + |\tilde{\nu}_1| + |\tilde{\nu}_2| + \dots + |\tilde{\nu}_m|) + 3 = k + 4 \leq n$ . Choose  $P_j \in \mathbb{N} \times \mathbb{N}$  such that  $|P_j| = j-1$  for  $j \in [n_0 + 1, n]$ . Define  $D \in \mathfrak{D}'$  as follows,

$$(5.3) \quad D = \{(0, 0)\} \cup \bigcup_{i=1}^m (E_{\tilde{\nu}_i} + T_i) \cup \bigcup_{j=n_0+1}^n \{P_j\},$$

where  $E_{\tilde{\nu}_i} + T_i$  denotes the set of points in  $\mathbb{N} \times \mathbb{N}$  obtained by adding each point in  $E_{\tilde{\nu}_i}$  by the translating vector  $T_i$ .

Now we prove the following claim.

*Claim.* Fix  $\nu \in \Pi_{d_2, k}$ . For any integer  $d'_2$  satisfying  $\#\nu \leq d'_2 \leq \binom{n}{2} - k - (\#\nu)$ , define  $d'_1 = \binom{n}{2} - k - d'_2$ . Then we can make choices of  $E_{\tilde{\nu}_i}$  and  $P_j$  in (5.3), such that the bi-degree of  $D$  is  $(d'_1, d'_2)$ , and the  $x$ -coordinates of the points in  $D$  are non-negative, i.e.  $D \in \mathfrak{D}$ .

*Proof of Claim.* We give the exact lower bound and upper bound for the  $y$ -degree of  $D$ , and shows that any integers between the lower bound and upper bound can be the  $y$ -degree of some  $D$ .

For the exact lower bound, we want to construct  $P_j$  and  $E_{\tilde{\nu}_i}$  such that their  $y$ -degrees are as small as possible. We let  $P_j = (j - 1, 0)$  and  $E_{\tilde{\nu}_i}$  be as follows:

$\tilde{\nu}_i$	$E_{\tilde{\nu}_i} \in \mathfrak{D}'$	$y$ -degree of $E_{\tilde{\nu}_i}$
$(1, 1, 1)$	$\{(-2, 2), (-1, 1), (0, 0)\}$	3
$(1, 1)$	$\{(-1, 1), (0, 0), (1, 1), (0, 2)\}$	2
$(1)$	$\{(-1, 1), (0, 0)\}$	1
$\begin{matrix} (v, w) \\ 2 \leq v \leq w \end{matrix}$	$\{(0, 0), (1, 0), \dots, (w - 1, 0)\} \cup \{(-1, 1), (w - v, 1)\}$	2
$\begin{matrix} (w) \\ w \geq 2 \end{matrix}$	$\{(-1, 1), (0, 0), (1, 0), \dots, (w - 1, 0)\}$	1

and denote the resulting  $D$  by  $D_{\min y}$ . Observe that the  $y$ -degree of  $E_{\tilde{\nu}_i}$  is equal to  $\#\tilde{\nu}_i$  for all  $\tilde{\nu}_i$  in the table, so the  $y$ -degree of  $D_{\min y}$  is  $\sum_{i=1}^m (\#\tilde{\nu}_i) = (\#\nu)$ .

For the exact upper bound, we need only to note that if  $D \in \mathfrak{D}_n$  can be constructed as (5.3), then the transpose of  $D$  (i.e. swap the  $x$  and  $y$  coordinates of each point in  $D$ ) can also be constructed as (5.3) for some choices of  $P_j$  and  $E_{\tilde{\nu}_i}$ . In particular, the transpose of  $D_{\min y}$ , denoted by  $D_{\max y}$ , can be constructed as (5.3). The  $y$ -degree of  $D_{\max y}$  is  $\binom{n}{2} - k - (\#\nu)$ , and is the maximal  $y$ -degree for all possible  $D \in \mathfrak{D}_n$  constructed as (5.3).

Finally, by moving an appropriate point of  $D$  to the north-west direction, the  $y$ -degree increases by 1, so every integer between  $\#\nu$  and  $\binom{n}{2} - k - (\#\nu)$  is the  $y$ -degree of some  $D$ . This completes the proof of Claim.  $\square$

Now by assumption  $d_2 \leq d_1$ ,  $d_1 + d_2 = \binom{n}{2} - k$ , and  $(\#\nu) \leq d_2$  since  $\nu$  is a partition of  $k$  into no more than  $d_2$  parts. Therefore  $(\#\nu) \leq d_2 \leq \binom{n}{2} - k - (\#\nu)$  and by the above claim  $d_2$  is the  $y$ -degree of some  $D \in \mathfrak{D}$  constructed as (5.3). Take this  $D$  and denote it by  $D_\nu$ . The bi-degree of  $D_\nu$  is  $(d_1, d_2)$ . Applying Lemma 2 (ii)(iii)(iv),

$$\varphi(D_\nu) = \prod_{i=1}^m \varphi(E_{\tilde{\nu}_i}),$$

hence by Lemma 22 (c),

$$\text{LM}(\varphi(D_\nu)) = \prod_{i=1}^m \text{LM}(\varphi(E_{\tilde{\nu}_i})) = \prod_{i=1}^m \rho_{\tilde{\nu}_i} = \rho_\nu.$$

CASE 2:  $\tilde{\nu}_m = (1, 1, 1)$ .



In this case,  $(\#\nu) = k = 3m$ . Choose  $D \in \mathfrak{D}$  to satisfy:  $|P_j| = j - 1$  for  $1 \leq j \leq n - 3m$ ,  $|P_{n-3m+3j-2}| = |P_{n-3m+3j-1}| = |P_{n-3m+3j}| = n = 3m + 3j - 3$ . By assumption,  $\nu \in \Pi_{d_2, k}$ , so  $d_2 \geq k$  in this case. It is straightforward to verify that we can choose such a  $D$  to have bi-degree  $(d_1, d_2)$ . This completes the proof of Theorem 23.  $\square$

*Proof of Theorem 24.* The proof is almost identical with the one of Theorem 23. We only need to modify the row  $\mu = (1, 1)$  in the table (5.2). Instead of using  $E_{(1,1)} \in \mathfrak{D}'$  (which contains 4 points), we use two elements  $E'_{(1,1)}$  and  $E''_{(1,1)}$  in  $D'$ , each of which contains 3 points.

$$\begin{aligned} E'_{(1,1)} &= \{(-a-1, a+1), (-a, a), (a+1, a)\}, \\ E''_{(1,1)} &= \{(-a-1, a+1), (-a, a), (a, a+1)\}. \end{aligned}$$

A simple computation shows

$$\varphi(E'_{(1,1)}) = \rho_2, \quad \varphi(E''_{(1,1)}) = -\rho_1^2 + \rho_2,$$

so

$$\varphi(E'_{(1,1)}) - \varphi(E''_{(1,1)}) = \rho_1^2.$$

Here we need to be cautious that the bi-degree of  $E'_{(1,1)}$  and  $E''_{(1,1)}$  are not the same. This will not bring any problem, since we can move points in other  $E_\mu$  to adjust the total bi-degree. Eventually, supposing that  $\ell$  is the integer that  $\tilde{\nu}_\ell = (1, 1)$ , we can construct  $D'_\nu, D''_\nu \in \mathfrak{D}$  both of bi-degree  $(d_1, d_2)$  such that

$$\varphi(D'_\nu) = \varphi(E'_{(1,1)}) \prod_{i \neq \ell} \varphi(E_{\tilde{\nu}_i}), \quad \varphi(D''_\nu) = \varphi(E''_{(1,1)}) \prod_{i \neq \ell} \varphi(E_{\tilde{\nu}_i}).$$

Then  $f := \Delta(D'_\nu) - \Delta(D''_\nu)$  satisfies  $\text{LM}(\varphi(f)) = \rho_\nu$ .

Now for each  $\nu \in \Pi_{d_2, k}$ , we can construct  $f_\nu$  such that  $\text{LM}(\varphi(f)) = \rho_\nu$ . If we write down the coefficient matrix for  $\varphi(f_\nu)$  with basis  $\{\rho_\mu\}_{\mu \in \Pi_k}$  arranged in decreasing order, we obtain a row echelon form with rank  $p(d_2, k)$ . So  $\dim M_{d_1, d_2} \geq p(d_2, k)$  by Lemma 27. Combining the upper bound obtained in Proposition 19, we conclude that  $\dim M_{d_1, d_2} = p(d_2, k)$ .  $\square$

## 6. THE CONDITION FOR THE EQUALITY $\dim M_{d_1, d_2} = p(d_2, k)$ TO HOLD

In Proposition 19 we showed the inequality  $\dim M_{d_1, d_2} \leq p(d_2, k)$ , then in Theorem 24 we showed that “=” holds for  $k \leq n - 3$ . In this section, we show that the condition  $k \leq n - 3$  is the best we can hope, in the sense of the following theorem.

**Theorem 35.** *Assume  $d_2 \leq d_1$ . Then  $\dim M_{d_1, d_2} \leq p(d_2, k)$ , and the equality holds if and only if “ $k \leq n - 3$ ”, or “ $k = n - 2$  and  $d_2 = 1$ ”, or “ $d_2 = 0$ ”.*

*Proof.* The inequality is proved in Proposition 19. Then we verify the equality  $\dim M_{d_1, d_2} = p(d_2, k)$  in the specified 3 cases. The case  $d_2 = 0$  is trivial since by definition  $p(0, k) = 0$  for  $k \geq 1$  and  $p(0, 0) = 1$ , we can check the equality directly. In the case  $k = n - 2$  and  $d_2 = 1$ ,  $\dim M_{d_1, d_2} = 1$  because  $\Delta(\{(0, 0), (0, 1), (1, 0), (2, 0), \dots, (n - 2, 0)\})$  forms a basis for  $M_{d_1, d_2}$ . The case  $k \leq n - 3$  is proved in Theorem 24.

Now assume  $d_2 \geq 2$ . We use the notation  $M_{d_1, d_2}^{(n)}$  to specify which  $n$  we are considering. By Proposition 19, it suffices to show that  $\dim M_{d_1, d_2}^{(n)} < \dim M_{d_1+n, d_2}^{(n+1)}$  for  $k = n - 2$ .

Using the condition  $d_1 \geq d_2$ , it is easy to check that  $(d_1 - n + 3) \geq 0$ . So both  $(d_1 - n + 3)$  and  $(d_2 - 2)$  are non-negative integers and  $(d_1 - n + 3) + (d_2 - 2) = \binom{n}{2} - k - n + 1 = \binom{n-2}{2}$ . We know that  $\dim M_{d_1-n+3, d_2-2}^{(n-2)} = 1$ . Let  $D^{(n-2)} \in \mathfrak{D}_{n-2}^{Catalan}$  be of bi-degree  $(d_1 - n + 3, d_2 - 2)$ . Define

$$D^{(n+1)} = \{(0, 0), (1, 0), (0, 2)\} \cup (D^{(n-2)} + (2, 0)).$$

Then  $D^{(n+1)} \in \mathfrak{D}_{n+1}^{Catalan}$  is of bi-degree  $(d_1 + n, d_2)$ . On the other hand, every  $D^{(n)} \in \mathfrak{D}_n^{Catalan}$  of bi-degree  $(d_1, d_2)$  determines an element

$$\{(0, 0)\} \cup (D^{(n)} + (1, 0))$$

in  $\mathfrak{D}_{n+1}^{Catalan}$  of bi-degree  $(d_1 + n, d_2)$ , which is distinct from  $D^{(n+1)}$ . Therefore

$$\dim M_{d_1, d_2}^{(n)} < \dim M_{d_1+n, d_2}^{(n+1)}.$$

□

**Remark 36.** For  $k = \binom{n}{2} - d_1 - d_2 = 0$ , we know  $\dim M_{d_1, d_2}^{(n)} = p(d_2, k) = 1$ . We give a straightforward construction of the element  $D \in \mathfrak{D}_n^{Catalan}$  of bi-degree  $(d_1, d_2)$  as follows. Let  $u = \lfloor (2n + 1 - \sqrt{4n^2 - 4n - 8d_1 + 9})/2 \rfloor$ ,  $i = nu - u(u + 1)/2 - d_1$ , define

$$(x_1, \dots, x_n) = (\underbrace{u-1, u-1, \dots, u-1}_i, \underbrace{u, u, \dots, u}_{n-u-i}, u-1, u-2, \dots, 1, 0),$$

$$y_i = \#\{j \mid i < j, x_j - x_i \in \{0, 1\}\}, \quad i = 1, \dots, n.$$

Then  $D = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$ .

□

**7.1. Table of the  $q, t$  Catalan number for  $n = 7$ .** The number located at the  $(i, j)$ -th coordinate is equal to the coefficient of  $q^i t^j$  in  $C_7(q, t)$ . The left at the bottom is the  $(0, 0)$ -th position.

[illegible]

Tables for  $n \leq 7$  can be found at F. Bergeron's website

`\protect\vrule width0pt\protect\href{http://bergeron.math.uqam.ca/n_fact_Conjecture/qt_c`

**7.2. Macaulay 2 code for computing  $\varphi$ .** For the convenience of the reader, we provide a Macaulay 2 code [11] for computing  $\varphi$  (function “phi” in the code) defined in §2.1, together with an example of computation

$$\varphi(\{(-1, 1), (0, 0), (0, 1), (0, 2), (1, 1)\}).$$

```
i1 : phi=(D)->(
      local R,n,k,sgn,s,total,t,bi,sumrho,prod;
      n=#D;
```

```

R=ZZ[r_0..r_n];
k=n*(n-1)//2; for i from 1 to n do
  k=k-D#(i-1)#0-D#(i-1)#1;
total=0;
scan(permutations(n),sigma->(
  sgn=1;
  for i from 0 to n do
    for j from i+1 to #sigma-1 do
      (if sigma#i>sigma#j then sgn=sgn*(-1));
  t=1;
  for i from 1 to n do
    (
      ai=D#(i-1)#0;bi=D#(i-1)#1;
      s=(sigma#(i-1)+1)-1-ai-bi;
      if (s<0) then (t=0;break) else
      if (bi==0) and (s>0) then (t=0;break) else
      if (bi==0) and (s==0) then (t=t*1) else
      if (bi==1) then (t=t*r_s) else
      (sumrho=0;
        scan(subsets(s+bi-1,bi-1),su->(
          prod=r_(su#0);
          for j from 2 to bi-1 do
            prod=prod*r_(su#(j-1)-su#(j-2)-1);
          prod=prod*r_(s+bi-2-su#(bi-2));
          sumrho=sumrho+prod;
        ));
      t=t*sumrho;
    );
  total=total+sgn*t;
)); --end of scan of sigma.
return sub((-1)^k*total,{r_0=>1});
)

```

o1 = phi

o1 : FunctionClosure

i2 : phi({(-1,1),(0,0),(0,1),(0,2),(1,1)})

$$o2 = -r_1^5 + 2r_1^3 r_2 + r_1^2 r_2^2 - 2r_1 r_3^2 - r_2 r_{23} + r_1 r_{14}$$

o2 : R

## REFERENCES

- [1] N. Bergeron and Z. Chen, Basis of Diagonally Alternating Harmonic Polynomials for low degree, arXiv: 0905.0377.
- [2] L. Carlitz, J. Riordan, Two element lattice permutation numbers and their  $q$ -generalization, Duke Math. J. 31 1964 371–388.
- [3] A. M. Garsia and J. Haglund, A positivity result in the theory of Macdonald polynomials, Proc. Natl. Acad. Sci. USA **98** (2001), no. 8, 4313–4316 (electronic).
- [4] A. M. Garsia and J. Haglund, A proof of the  $q, t$ -Catalan positivity conjecture. LaCIM 2000 Conference on Combinatorics, Computer Science and Applications (Montreal, QC). Discrete Math. 256 (2002), no. 3, 677–717.
- [5] A. M. Garsia and M. Haiman, A Remarkable  $q; t$ -Catalan sequence and  $q$ -Lagrange inversion, J. Algebraic Combin. **5** (1996), 191–244.
- [6] M. Haiman, Hilbert schemes, polygraphs and the Macdonald positivity conjecture, J. Amer. Math. Soc. **14** (2001), no. 4, 941–1006.
- [7] M. Haiman, Vanishing theorems and character formulas for the Hilbert scheme of points in the plane, Invent. Math. **149** (2002), no. 2, 371–407.
- [8] M. Haiman, Commutative algebra of  $n$  points in the plane, With an appendix by Ezra Miller. Math. Sci. Res. Inst. Publ., 51, Trends in commutative algebra, 153–180, Cambridge Univ. Press, Cambridge, 2004.
- [9] G.H. Hardy, Ramanujan: twelve lectures on subjects suggested by his life and work, Chelsea Publishing Company, New York 1959.
- [10] K. Lee, L. Li, Notes on a minimal set of generators for the radical ideal defining the diagonal locus of  $(\mathbb{C}^2)^n$ , Arxiv math 0901.1176.
- [11] D. Grayson, M. Stillman, Macaulay 2, a software system for research in algebraic geometry, available at <http://www.math.uiuc.edu/Macaulay2/>.
- [12] A. Woo, private communication.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907

*E-mail address:* `kyung1@purdue.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA CHAMPAIGN, URBANA, IL 61801

*E-mail address:* `llpku@math.uiuc.edu`